From KPZ Scaling to long-lived Solitons in the classical Heisenberg Chain

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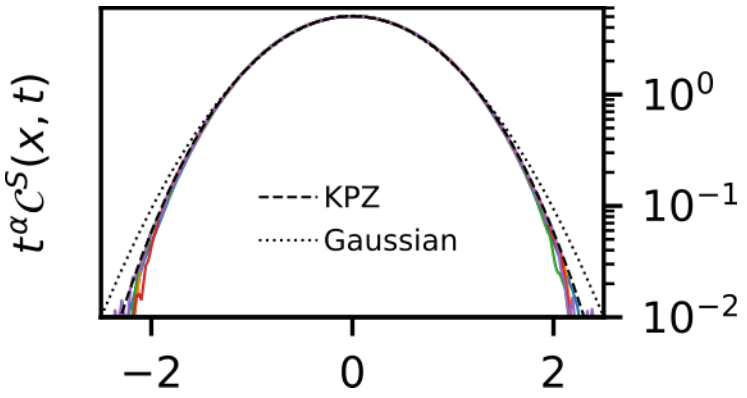
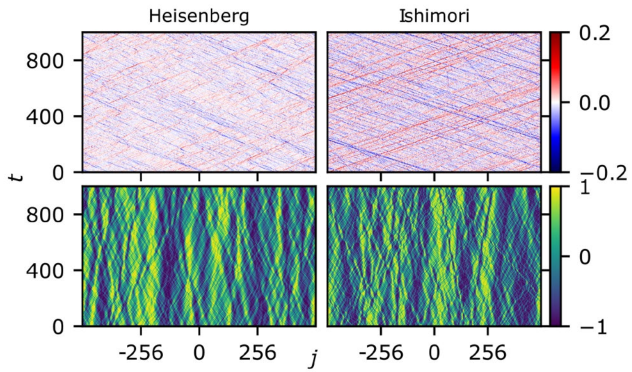




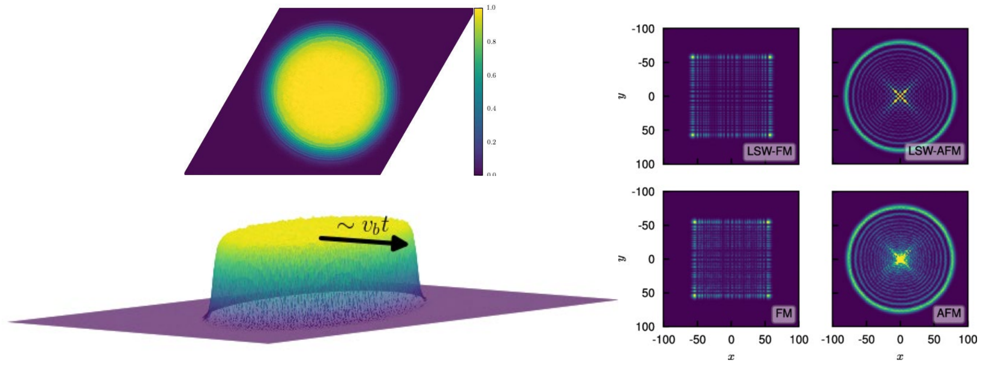
PCS International Workshop on Condensed Matter Solitons

Phys. Rev. B 105, L100403 (2022)

arxiv (soon)

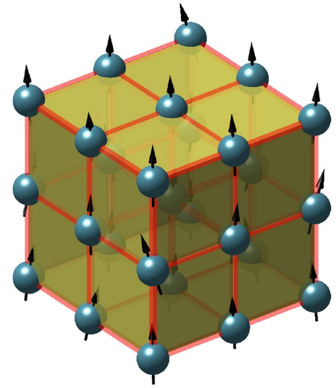
Recent Surprises in Classical Many-Body Spin Dynamics

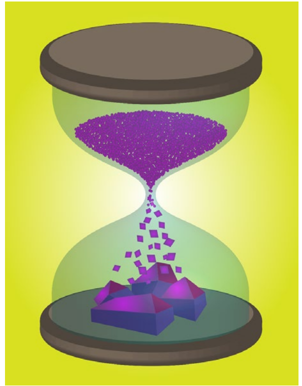
Classical Chaos & OTOC's



Das et al, PRL. 121, 024101; **TB** et al. , PRL 121, 250602; PRB 103, 174302

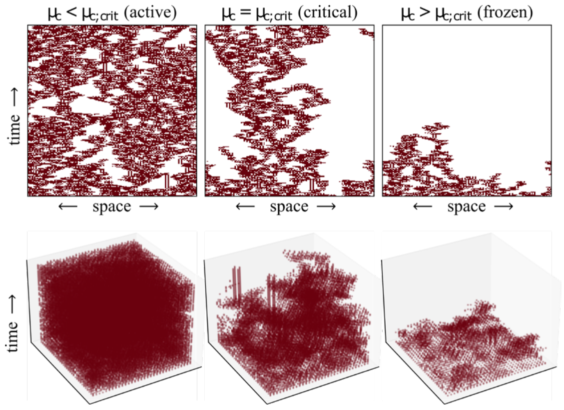
Prethermal Time-Crystals





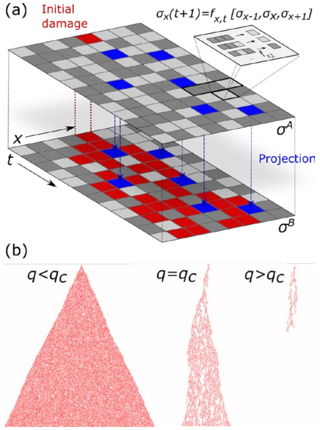
N.Yao PRL 127, 140603; A.Pizzi, PRL 127, 140602; G.Engelardt, *Physics* 14, 132

Many-Body Chaos & Kinetic Constraints



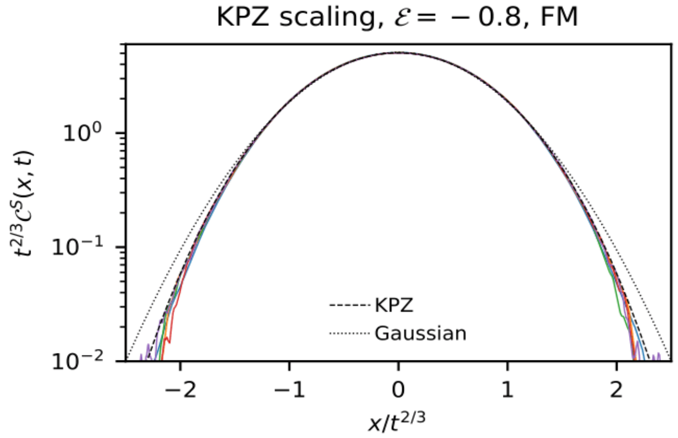
A. Deger, 2206.07724v1, arXiv:2202.11726v1

Measurement Induced Phases



J. Wilsher, arXiv:2203.11303;

Hydrodynamics/Integrability/KPZ/Solitons



Das et al, Phys. Rev. E 100, 042116; **A.McRoberts, et al, PRB 105, L100403**

Non-Equilibrium Dynamics

- Many-body systems are complex
- Universality central to our understanding of nature
- can occur in long-time non-equilibrium dynamics

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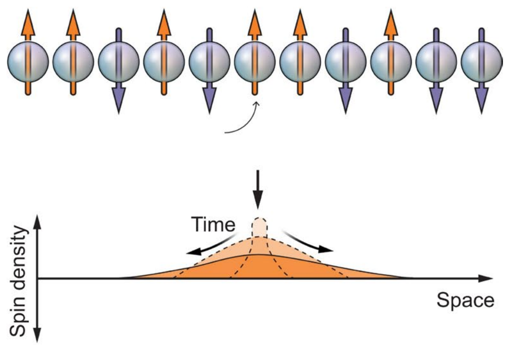

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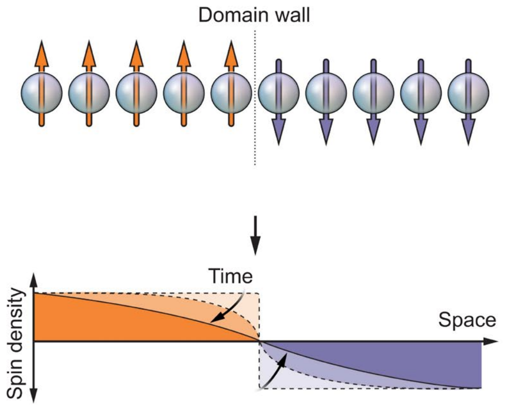


V. ALTOUNIAN AND N. CARY/*SCIENCE*
science.abn6376

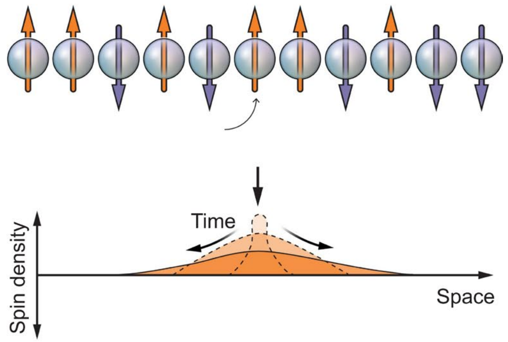
[1] M. K. Joshi et al., Science 376, 720 (2022) [2] D. Wei et al., Science 376, 716 (2022)

Non-Equilibrium Dynamics

- Many-body systems are complex
- Universality central to our understanding of nature
- can occur in long-time non-equilibrium dynamics

- Most well known non-trivial: **Kardar-Parisi-Zhang (KPZ)** Universality class [3]
  - Originally for stochastic surface growth
  - Describes many other scenarios, SEP, RMT, q XXZ chain, one-dimensional Bose gas, FPU chain, Ishimori chain

[1]



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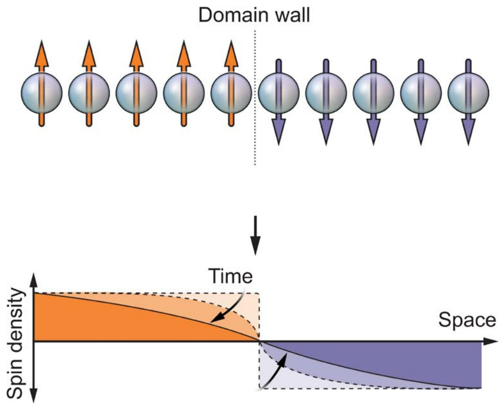

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[1] M. K. Joshi et al., Science 376, 720 (2022) [2] D. Wei et al., Science 376, 716 (2022) [3] M.Kardar,G.Parisi, Y-C Zhang (1986)

Background – Diffusive Hydrodynamics

- Hydrodynamics – concerned with the physics of “slow-modes” – usually locally conserved quantities (“charges”)

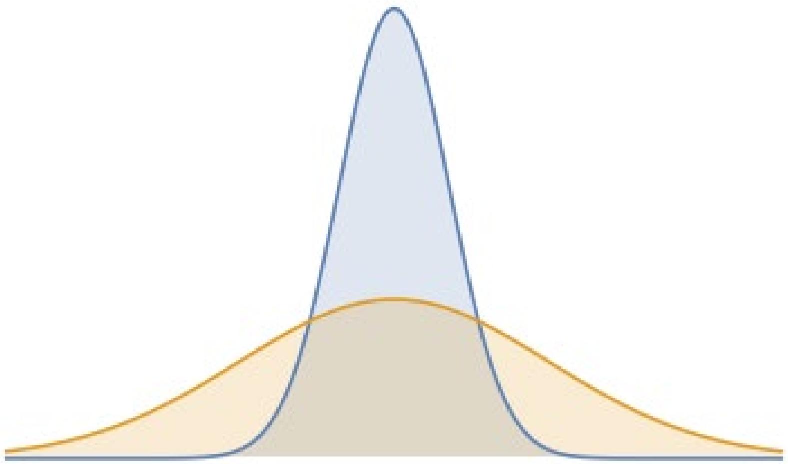
- $\partial_t \rho + \partial_x j_x = 0$
- $j_x = -D\, \partial_x \rho + O\big((\partial j)^2, \partial^2 j\big)$

$$\Rightarrow \qquad \partial_t q + \frac{D}{2}\partial_x^2 q = 0$$

- Ordinary diffusion follows

$$\mathcal{C}(x,t) = \langle q(x,t) q(0,0)\rangle - \langle q\rangle^2 \qquad \mathcal{C}(x,t) = \frac{\kappa}{(Dt)^\alpha} \exp\left[-\left(\frac{x}{(Dt)^\alpha}\right)^2\right], \quad \alpha = 1/2$$

- More generally: $\mathcal{C}(x,t) \sim t^{-\alpha}\mathcal{F}(t^{-\alpha}x)$



Departures (non comprehensive)
- Integrable Models <-> Infinite number of conservation Laws
- Higher-Order Derivatives, e.g. Fracton Models

Background – Nonlinear Fluctuating Hydrodynamics

- One of the main analytic explanations of KPZ scaling
- Applies to fluids with three conserved quantities: mass, energy, and momentum
- Explains anomalous scaling in easy-plane XXZ chain, with conserved magnetisation, energy, and azimuthal phase difference [1,2]
- Momentum conservation necessary to generate non-trivial Euler hydrodynamics
- Adding non-linear term and noise to diffusion equation yields the KPZ equation

[1] A. Das, K. Damle, A. Dhar, D. A. Huse, M. Kulkarni, C. B. Mendl, & H. Spohn (2019), [2] C. Mendl, H. Spohn (2015)

Background – Generalised Hydrodynamics [1]

- Analytic theory for KPZ/anomalous scaling [1,2], e.g. for quantum XXZ [3]
- Applies to integrable systems – Toda chain, Lieb-Liniger model, Ishimori chain, etc.
- Generalised Gibbs ensemble
- Relies on well-defined quasiparticles/integrability/infinite number of conserved charges
- Allows description in terms of scattering map

[1] B. Doyon (2020) [2] J.Nardis, D.Bernard, B. Doyon (2019) [3] S.Gopalakrishnan, R. Vasseur (2019); M.Ljubotina, M.Znidaric, T. Prosen (2019)

Outline

- Part I (KPZ in Heisenberg Chain)
  - Equilibrium Hydrodynamics
  - Equilibration Dynamics
- Part II (Solitons in Heisenberg Chain)
  - Single Soliton Solutions
  - Soliton Scattering
  - Solitons in Thermal States
- Conclusions

Classical Heisenberg Chain

- One of the simplest interacting lattice models: $\mathcal{H} = -J \sum_j \boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1}$

- Dynamics given by Landau-Lifshitz equation of motion:

$$\dot{\boldsymbol{S}}_j = J \boldsymbol{S}_j \times (\boldsymbol{S}_{j-1} + \boldsymbol{S}_{j+1})$$

- Obtained from the spin-commutator algebra:

$$[S^\mu, S^\nu] = i\epsilon^{\mu\nu\lambda} S^\lambda \mapsto \{S^\mu, S^\nu\} = \epsilon^{\mu\nu\lambda} S^\lambda$$

- Connected correlators of spin and energy:

$$\mathcal{C}^S(j,t) = \langle \boldsymbol{S}_j(t) \cdot \boldsymbol{S}_0(0) \rangle$$

$$\mathcal{C}^E(j,t) = \langle E_j(t) E_0(0) \rangle - \langle \mathcal{E}^2 \rangle, \quad E_j = -J \boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1}$$

Hydrodynamics of the Heisenberg Chain

- Nature of hydrodynamics debated for a long time
- Diffusive [1]
- Non-diffusive [2]
- Recent prediction of logarithmic anomaly [3]: $D(t) \propto \log(t)^{4/3}$
- Refuted by a recent theory of non-abelian hydrodynamics [4], which predicts ordinary diffusion, with $\sqrt{t}$ - corrections $D(t)^{-1} = \frac{1}{D\sqrt{t}} + \frac{\Lambda}{t}$

[1] PRL 63, 812 (1989),PRB 42, 8214 (1990), PRL 70, 248 (1993), PRB 80,115104 (2009), PRB 87, 075133 (2013), PRL 121, 024101 (2018), PRB 101, 041411 (2020), PRB 101,121106 (2020)

[2] J. Appl. Phys. 75, 6751 (1994), PRB 99, 140301 (2019), PRB 101,121106 (2020), PRL 124, 210605 (2020)

[3] J. de Nardis, M. Medenjak, C. Karrasch, & E. Ilievski (2020)

[4] P. Glorioso, L. Delacrétaz, X. Chen, R. Nandkishore, (2021)

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- Our contribution:
  - larger systems, longer times, large temperature range
  - Qualitative difference between ferromagnet and antiferromagnet
  - **KPZ scaling**
  - **Solitons**

[1] PRL 63, 812 (1989),PRB 42, 8214 (1990), PRL 70, 248 (1993), PRB 80,115104 (2009), PRB 87, 075133 (2013), PRL 121, 024101 (2018), PRB 101, 041411 (2020), PRB 101,121106 (2020)
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[4] P. Glorioso, L. Delacrétaz, X. Chen, R. Nandkishore, (2021)

Numerical Methods

- Systems of $\sim$10,000 spins
- Sample $\sim$ 20,000 states from canonical ensemble at given temperature/energy via Monte Carlo
- Numerically solve equations of motion for up to $t = 10^5$
- Energy and spin-norm are conserved to machine precision, m up to $\sim 10^{-5}$

Observables

- Ensemble-averaged connected correlation functions of conserved densities

$$\mathcal{C}^E(j,t) = \langle E_j(t)E_0(0)\rangle - \mathcal{E}^2, \quad \mathcal{E} = \frac{1}{L}\sum_j E_j(t) \qquad \mathcal{C}^S(j,t) = \langle \boldsymbol{S}_j(t)\cdot\boldsymbol{S}_0(0)\rangle$$

- Hydrodynamic scaling: $\mathcal{C}(x,t) \sim t^{-\alpha}\mathcal{F}(x/t^{\alpha})$

- Scaling exponent extracted from the autocorrelation function: $\mathcal{A}(t) = \mathcal{C}(0,t) \sim t^{-\alpha}$

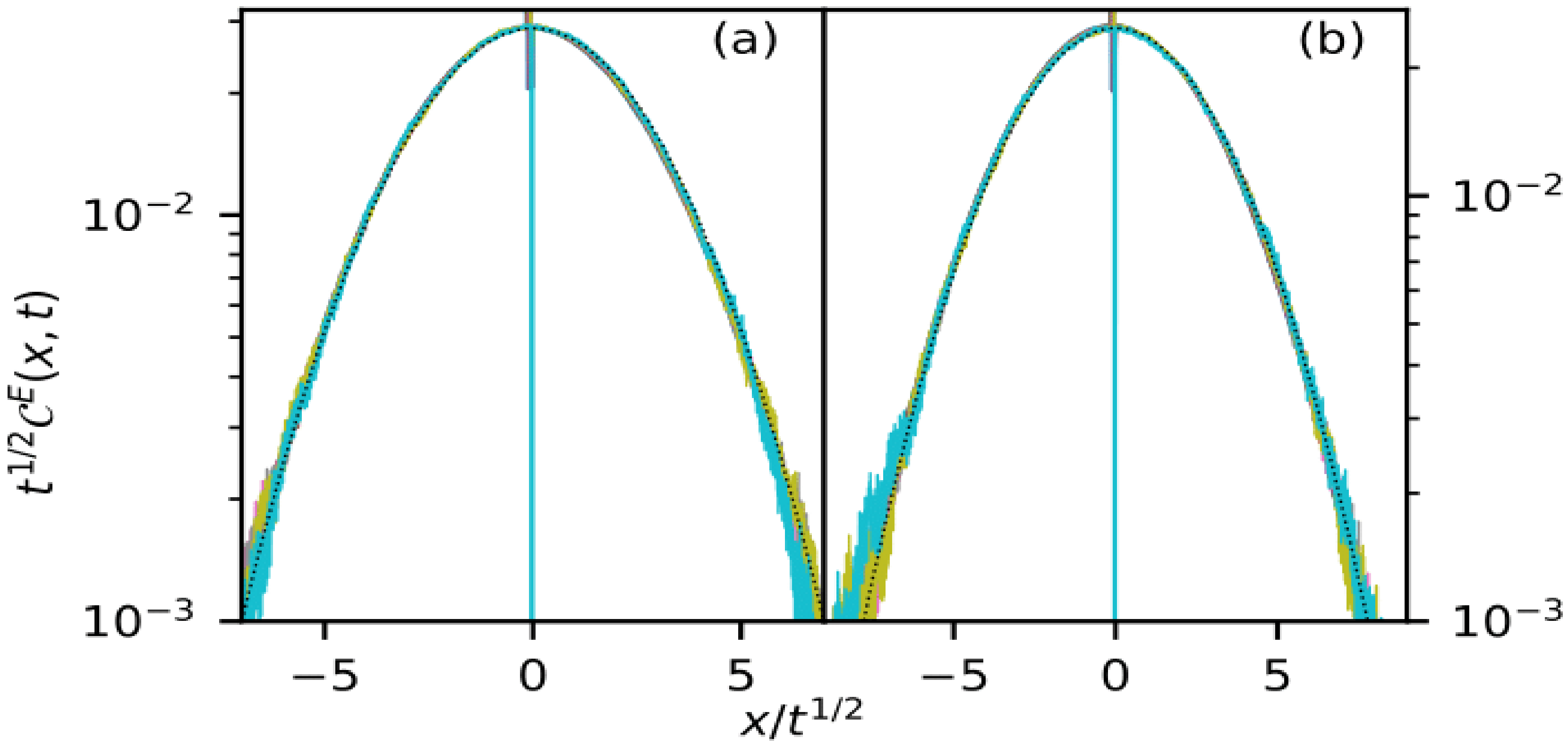
- Can also extract crossover timescale by assuming: $\mathcal{A}(t) = (Dt)^{-1/2} + \Lambda t^{-1}$

- Effective exponent given by: $\alpha_{\text{eff}}(t) = -\frac{d\log(\mathcal{A})}{d\log(t)}$

Equilibrium Hydrodynamics – Energy

- Energy correlations are diffusive

$$\mathcal{C}(x,t) = \frac{\kappa}{(Dt)^\alpha} \exp\left[-\left(\frac{x}{(Dt)^\alpha}\right)^2\right], \quad \alpha = 1/2$$



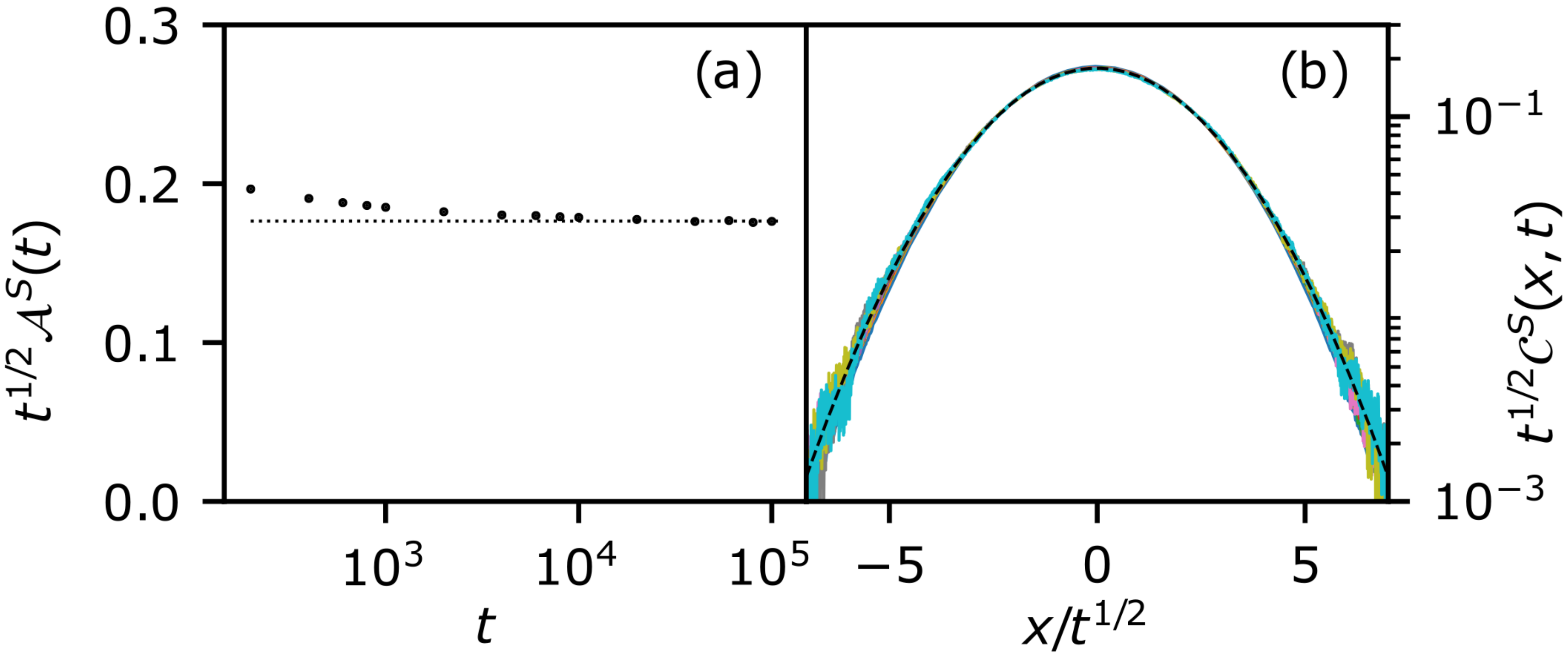
(a) AFM, E = -0.5
(b) FM, E = -0.5

Equilibrium Hydrodynamics – Spin – T = ∞

- Perfectly diffusive at infinite T
- No evidence of log-correction

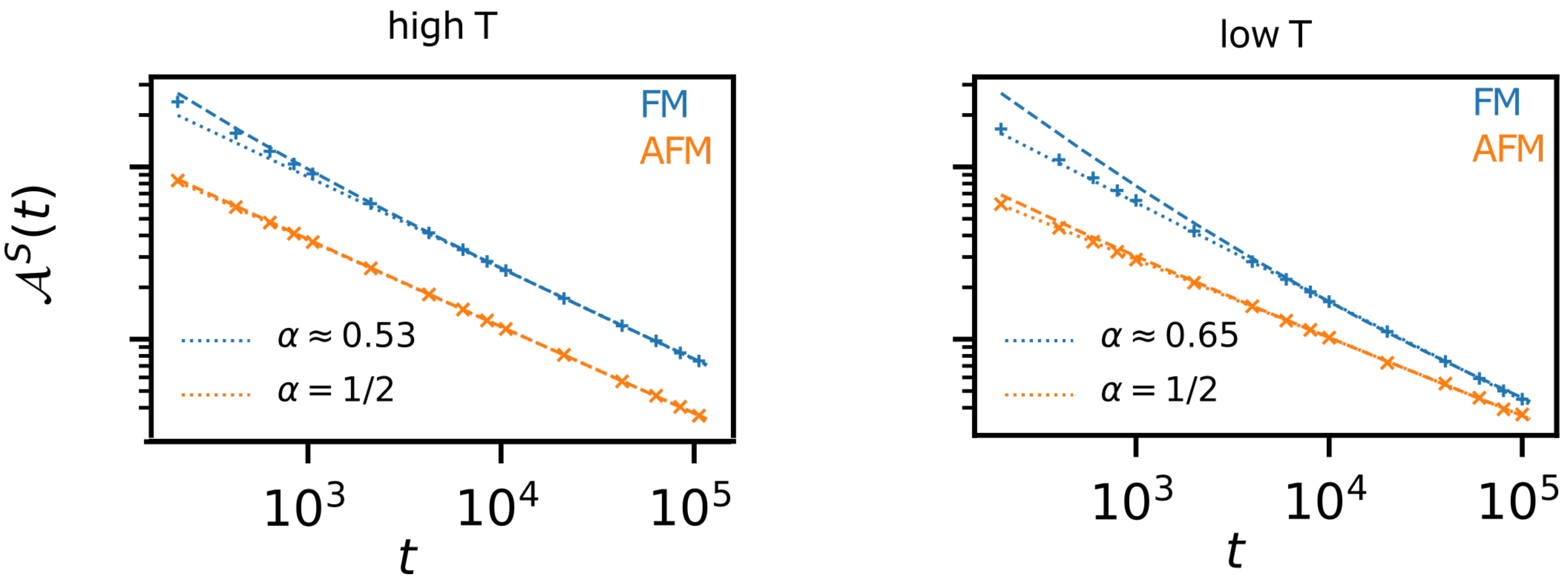
$$\mathcal{C}(x,t) = \frac{\kappa}{(Dt)^{\alpha}} \exp\left[-\left(\frac{x}{(Dt)^{\alpha}}\right)^2\right], \quad \alpha = 1/2$$

$$\mathcal{A}(t) = \mathcal{C}(0,t) \sim t^{-\alpha}$$

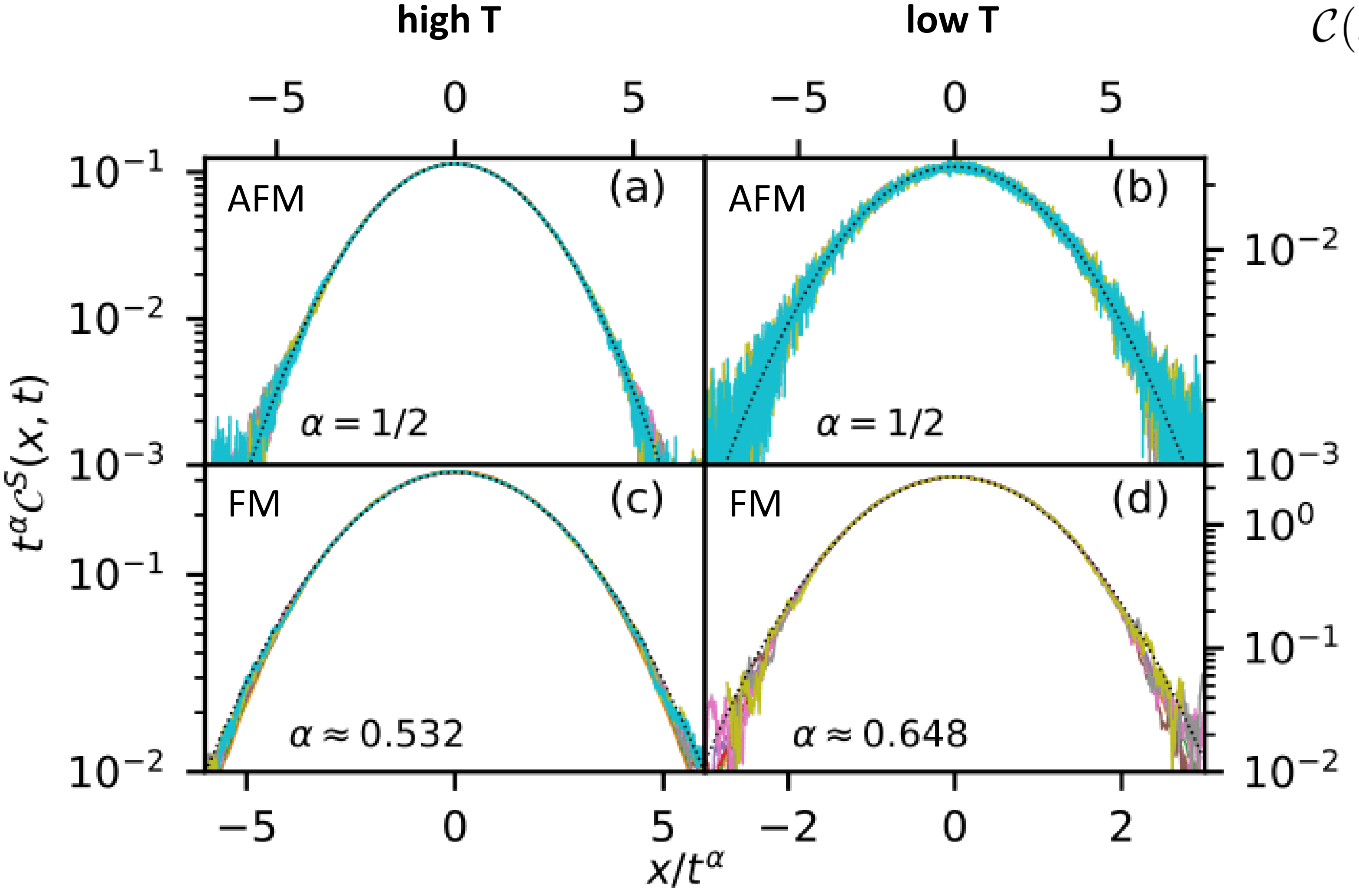


Equilibrium Hydrodynamics – Spin

- FM and AFM spin correlation show distinct dynamics
- FM spin correlations are anomalous (superdiffusive); AFM spin correlations are diffusive



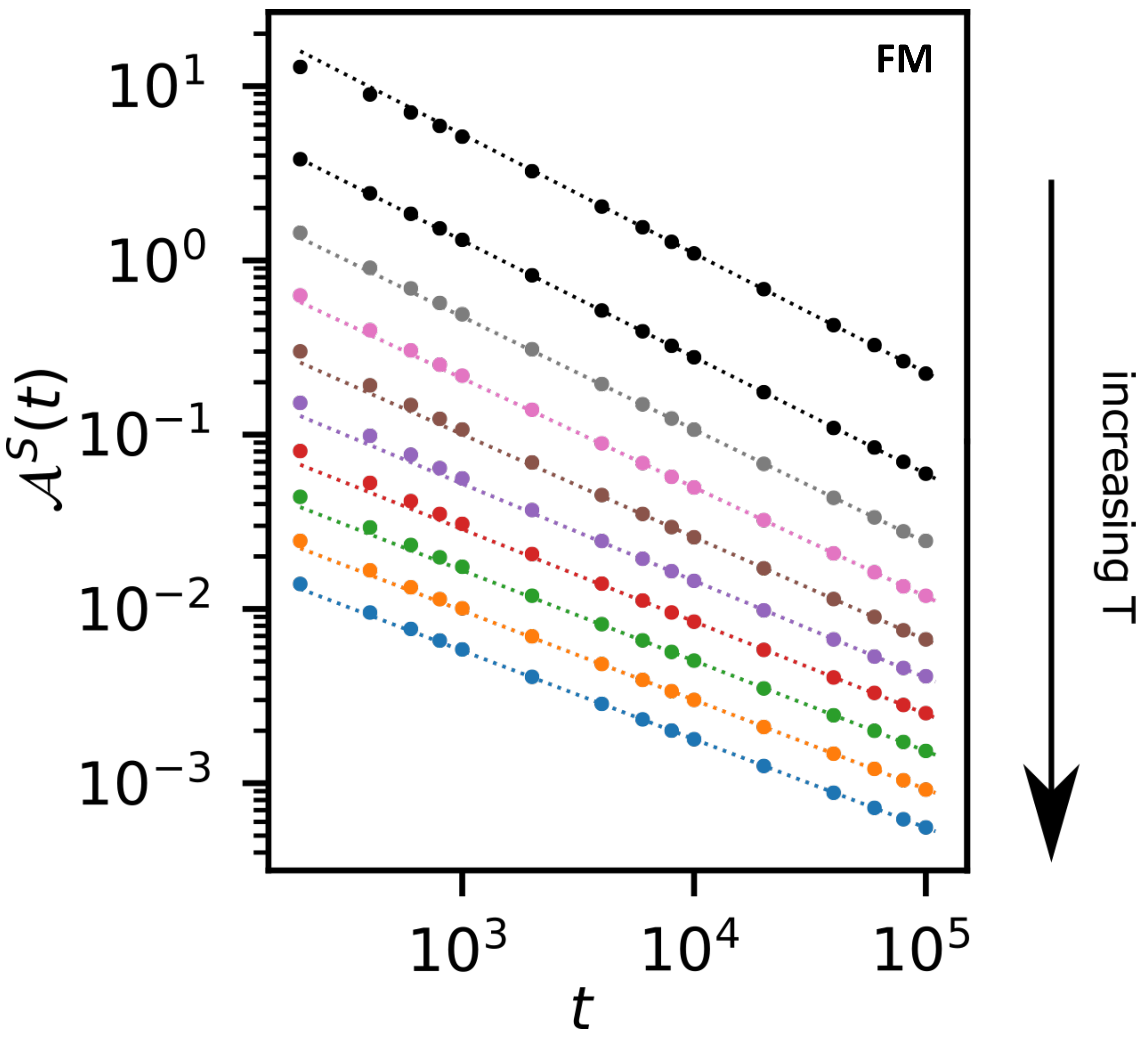
Equilibrium Hydrodynamics – Spin



$$\mathcal{C}(x,t) \approx \frac{\kappa}{(Dt)^\alpha} \exp\left[-\left(\frac{x}{(Dt)^\alpha}\right)^2\right]$$

(a) AFM, E = -0.3
(b) AFM, E = -0.7
(c) FM, E = -0.3
(d) FM, E = -0.7

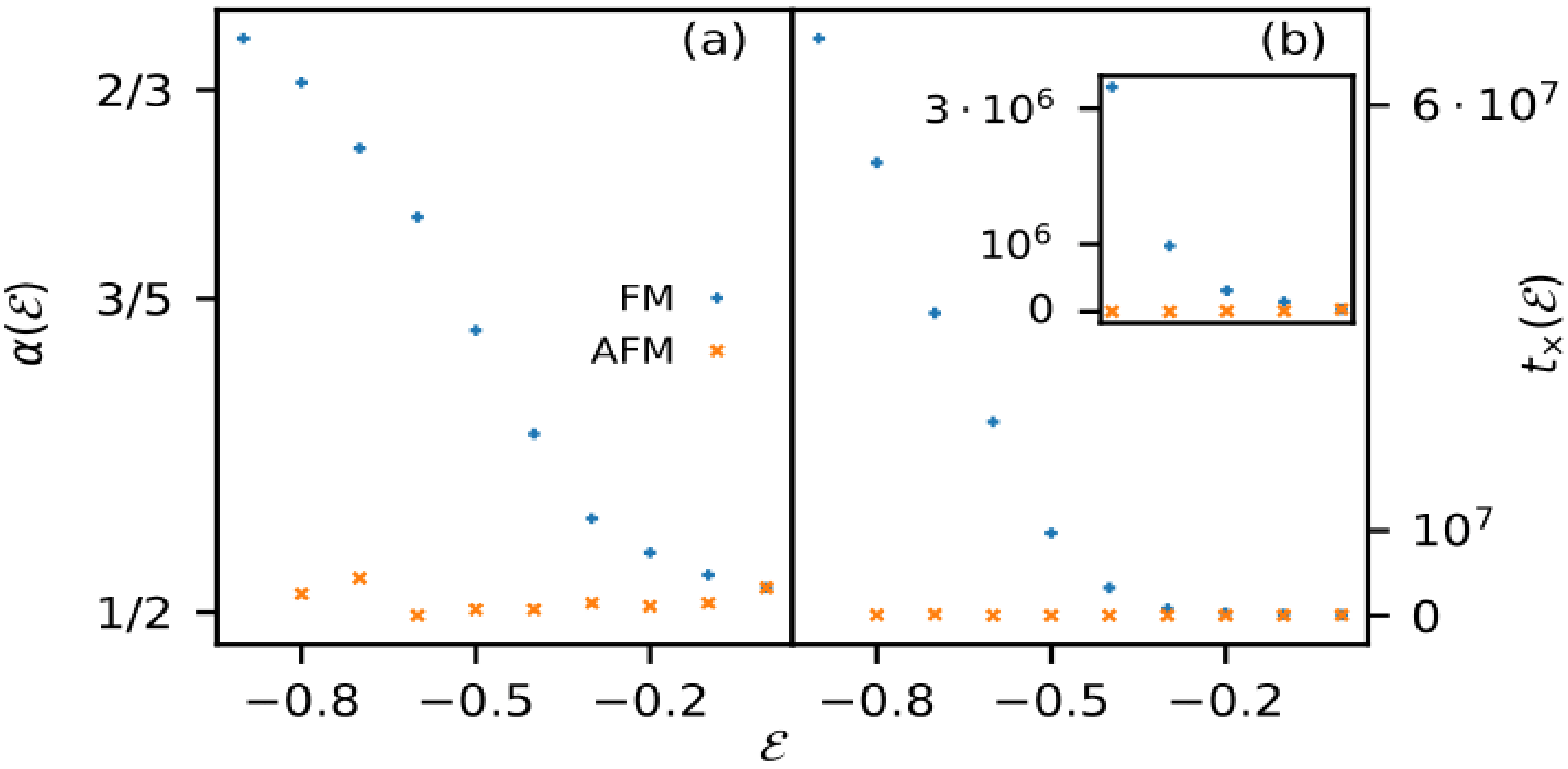
Equilibrium Hydrodynamics – Spin – FM



- Perfect straight line fit at T=0.2
- Close to KPZ exponent ($\alpha = \frac{2}{3}$)

Equilibrium Hydrodynamics – Spin

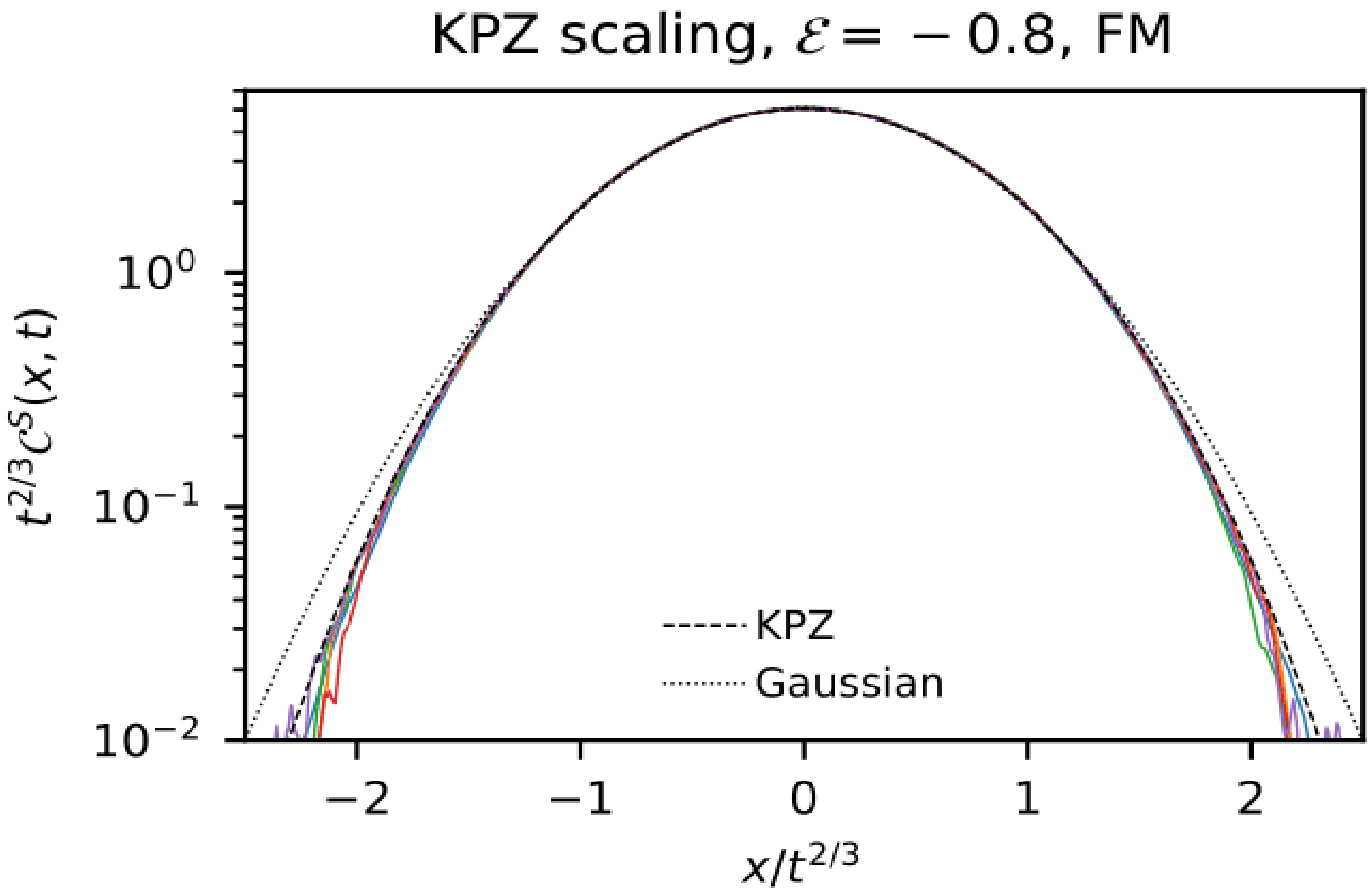
- FM appears anomalous at low T
- Crossover times (if real) increase rapidly



Equilibrium Hydrodynamics – Spin – KPZ Scaling

- FM Spin Correlations are KPZ

$$\mathcal{C}^S(x,t) \sim t^{-2/3} f_{\mathrm{KPZ}}(x/t^{2/3})$$



Equilibration After a Quench

- Establishing global equilibrium is related to equilibrium hydrodynamics
- Begin with the XY chain: $\mathcal{H} = -J \sum_j \cos(\phi_j - \phi_{j+1})$
- (same as the Heisenberg chain, but with every spin confined to the plane)
- Initial states drawn from a canonical ensemble of the XY chain
- At t > 0, dynamics is generated by the Heisenberg chain
- How does the system relax towards its (isotropic) equilibrium?

Equilibration - Observables

- Measures of the anisotropy:

$$Q^\mu(t) = \langle (S_i^\mu(t))^2 \rangle \qquad E^\mu(t) = \langle S_i^\mu(t) S_{i+1}^\mu(t) \rangle$$

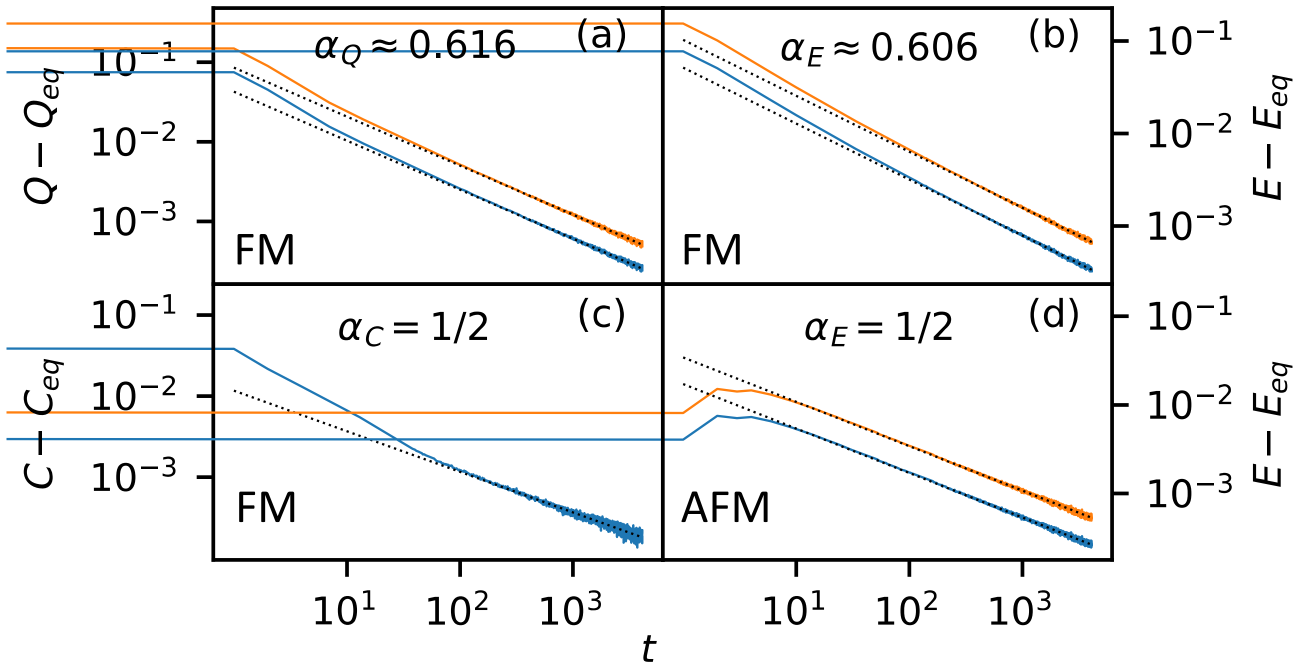
- Measure of the energy distribution:

$$C(t) = \mathrm{var}_i E_i / T^2 \qquad \text{cf. } C = \frac{\langle E^2 \rangle - \langle E \rangle^2}{T^2}$$

- Equilibration expected to follow power-law:

$$|\mathcal{O}(t) - \mathcal{O}_{\mathrm{eq}}| = \lambda t^{-\alpha}$$

Results – Equilibration



E = -0.5 in all cases

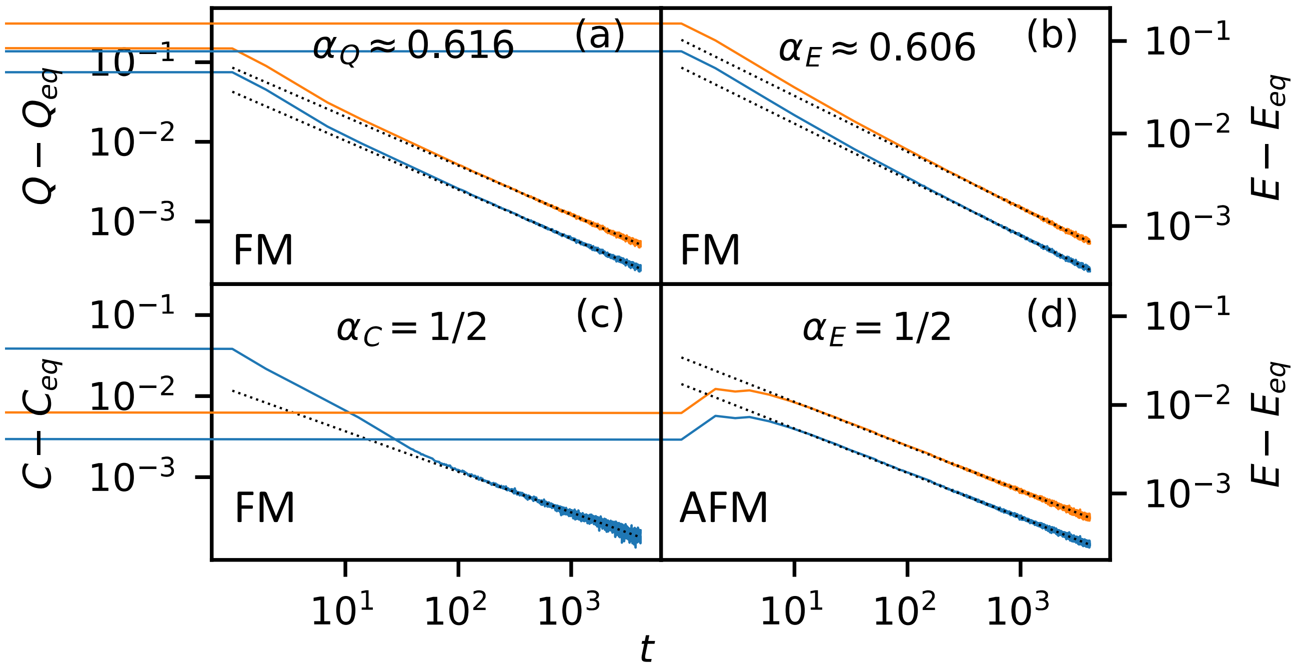
(a) Equilibration of Q in the FM

(b) Equilibration of E in the FM

(c) Equilibration of C in the FM

(d) Equilibration of E in the AFM

Results – Equilibration



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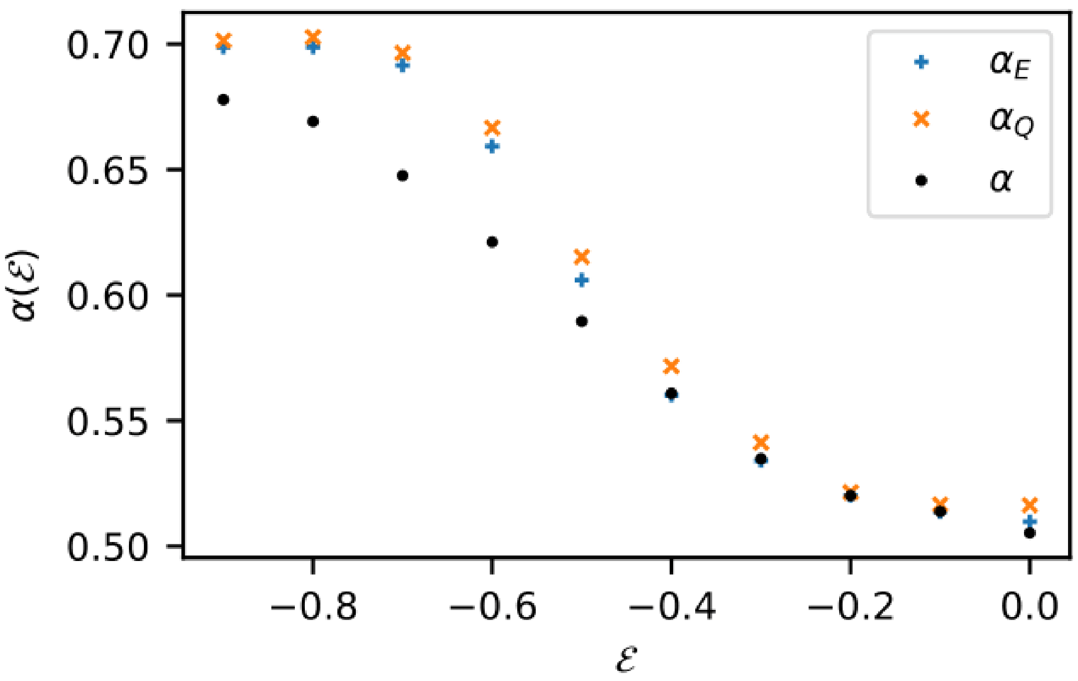
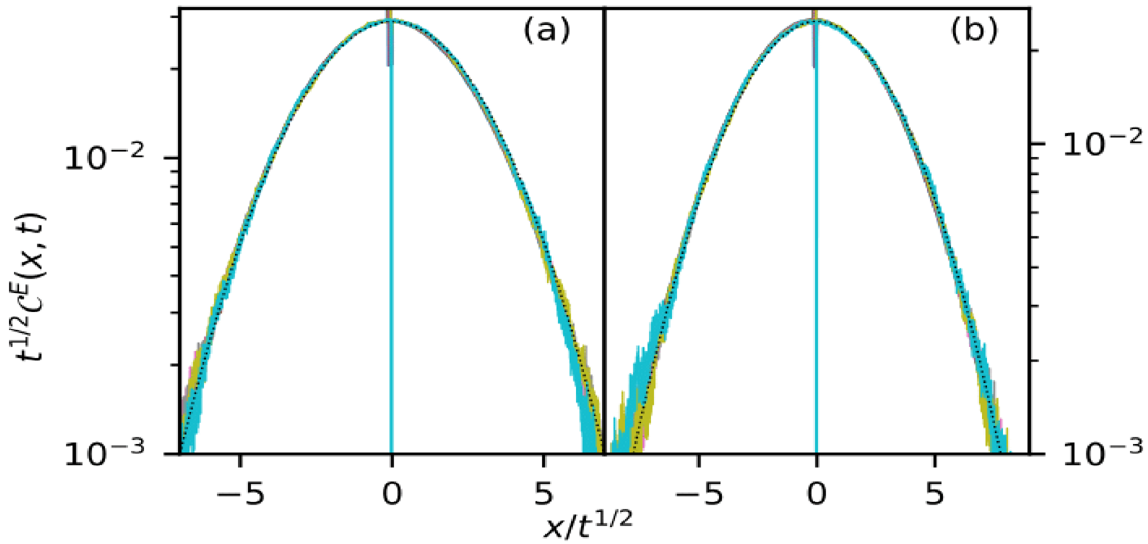
(c) Equilibration of C in the FM
(d) Equilibration of E in the AFM



Comparison of equilibrium and equilibration exponents of the FM

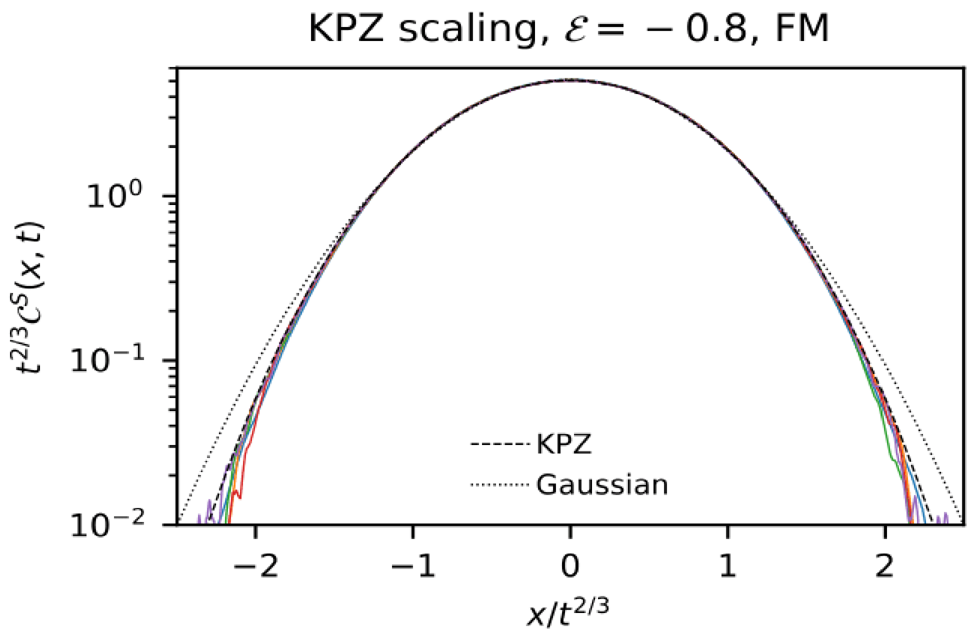
Summary – Equilibrium Hydrodynamics – Energy

- Energy correlations are diffusive
- Finite-time ballistic correlations at low temperature, crossing over to diffusion at longer times
- Ferromagnet and Antiferromagnet have the same scaling of energy correlations



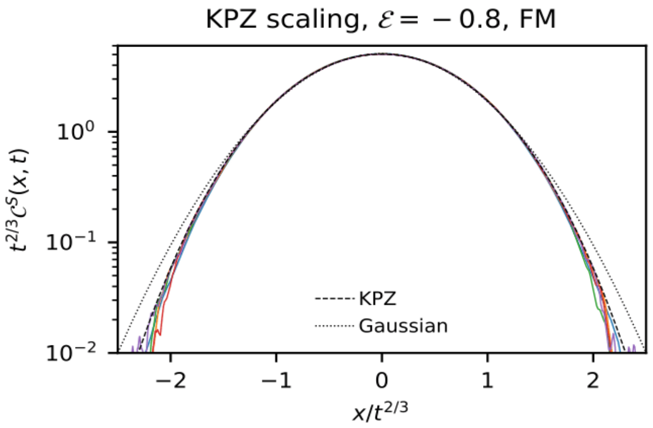
Summary – Equilibrium Hydrodynamics – Spin

- Spin correlations are anomalous (superdiffusive) for the ferromagnetic chain
- True even at high temperatures, with correlation length ∼ a few lattice sites
- Long-lasting KPZ scaling at low temperature
- Antiferromagnetic chain is diffusive



Discussion and Conclusions

- Classical Heisenberg chain is one of the simplest dynamical models of magnetism
- Hides a rich regime of (at least very long-lived) superdiffusive spin hydrodynamics; but only in the FM
- Very strong numerical evidence for (at least long-lived) KPZ scaling at low temperature in the FM
- Possible explanation – proximity to integrable continuum model?
- but: occurs in regime of short correlation lengths
- True explanation may hide in proximity to integrable lattice spin-chains



Part II: Solitons

Solitons in classical spin models

Integrable Models

- Continuum Landau Lifshitz [1]
- Ishimori/Integrable Landau-Lifshitz Chain [2]

[1] H.C. Fogedby J.Phys.Math.A. (1980) [2] Yuji Ishimori, J. Phys. Soc. Jpn. 51

Solitons in classical spin models

Integrable Models

- Continuum Landau Lifshitz [1]
- Ishimori/Integrable Landau-Lifshitz Chain [2]

Heisenberg Chain

- Non-integrable
- Continuum approach: Modulated spin waves [3]
- Some exact solutions [4]

[1] H.C. Fogedby J.Phys.Math.A. (1980) [2] Yuji Ishimori, J. Phys. Soc. Jpn. 51, [3] H-J Schmidt, C. Schroeder, M Luban J.Phys.Cond.Mat (2011) [4] M.Lakshmanan Phi.Trans.R.Soc. A (2011)

The Integrable Ishimori Chain [1]

$$\mathcal{H} = -2J \sum_i \log\left(\frac{1 + \boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1}}{2}\right)$$

[1] Yuji Ishimori, J. Phys. Soc. Jpn. 51, [2] N.Theodorakopoulos, Physics Letters A, Volume 130
[3] Das et al, Phys. Rev. E 100, 042116, Journal of Statistical Physics volume 180, [4] Roy et al., arXiv:2205.03858

The Integrable Ishimori Chain [1]

$$\mathcal{H} = -2J \sum_i \log\left(\frac{1 + \boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1}}{2}\right)$$

- Integrable via inverse-scattering and mapping to discrete non-linear Schroedinger equation [1]
- Allows soliton solutions [1]
- 2-soliton phase-shifts [2]
- full thermodynamics of soliton gas [2]

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- full thermodynamics of soliton gas [2]

- Infinite number of conserved charges, e.g. $\tau_i = \dfrac{\boldsymbol{S}_i \cdot (\boldsymbol{S}_{i+1} \times \boldsymbol{S}_{i-1})}{(1 + \boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1})(1 + \boldsymbol{S}_i \cdot \boldsymbol{S}_{i-1})}$

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The Integrable Ishimori Chain [1]

$$\mathcal{H} = -2J \sum_i \log\left(\frac{1 + \boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1}}{2}\right)$$

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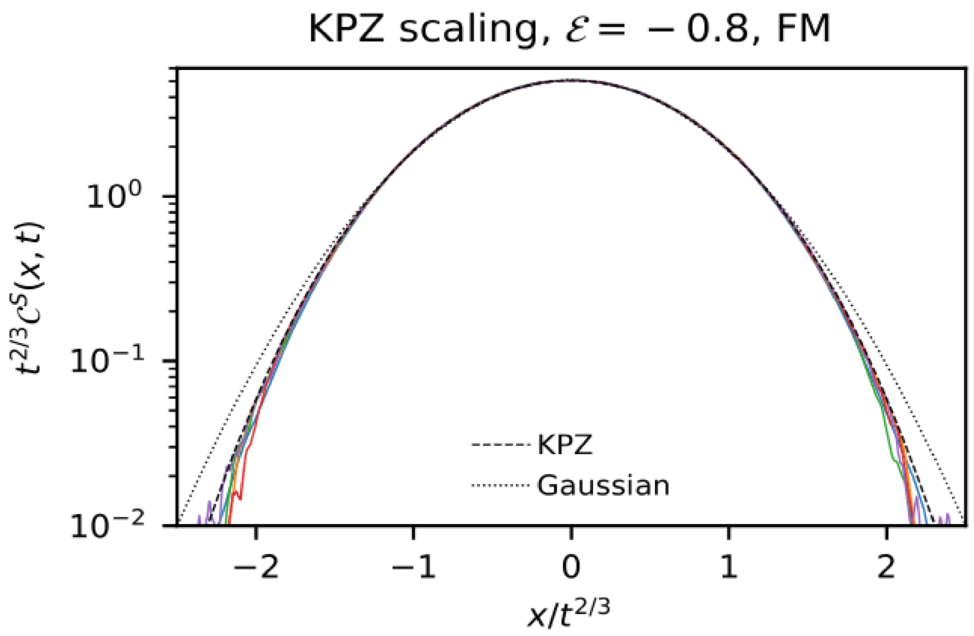
- Infinite number of conserved charges, e.g. $\tau_i = \dfrac{\boldsymbol{S}_i \cdot (\boldsymbol{S}_{i+1} \times \boldsymbol{S}_{i-1})}{(1 + \boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1})(1 + \boldsymbol{S}_i \cdot \boldsymbol{S}_{i-1})}$
- Shows KPZ scaling [3]
- KPZ is robust to (symmetry-preserving) perturbations [4]

[1] Yuji Ishimori, J. Phys. Soc. Jpn. 51, [2] N.Theodorakopoulos, Physics Letters A, Volume 130
[3] Das et al, Phys. Rev. E 100, 042116, Journal of Statistical Physics volume 180, [4] Roy et al., arXiv:2205.03858

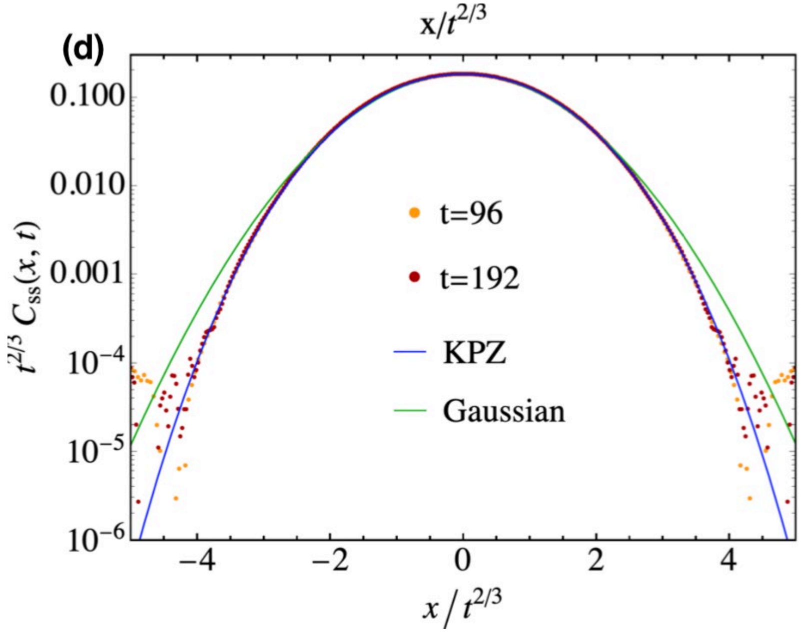
KPZ in Ishimori Chain vs Heisenberg

$$\mathcal{H} = -J \sum_i (\boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1})$$

$$\mathcal{H} = -2J \sum_i \log\left(\frac{1 + \boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1}}{2}\right)$$



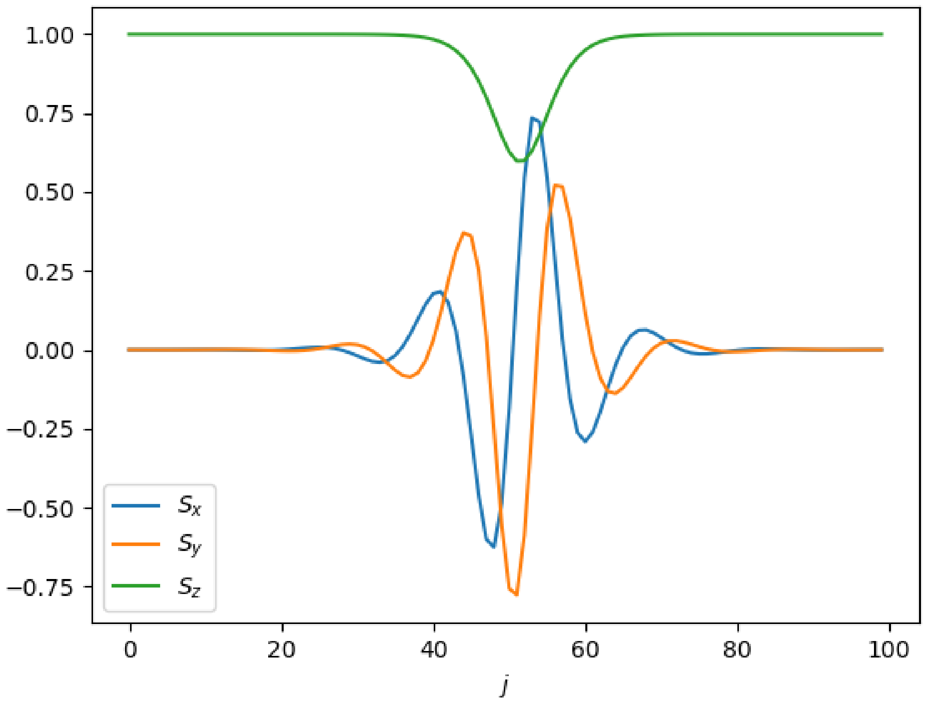
McRoberts et al, Phys. Rev. B 105, L100403



Das et al., Phys. Rev. E 100, 042116

The Integrable Ishimori Chain [1]

1-Soliton Solutions



[1] Yuji Ishimor, J. Phys. Soc. Jpn. 51

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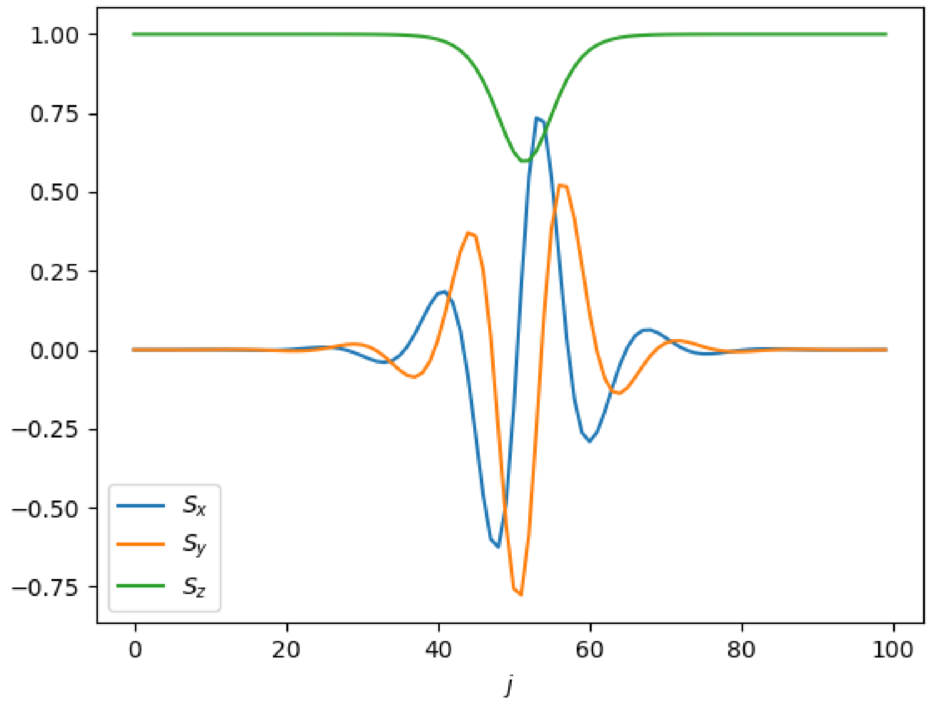
$$S_i^x(t) = \frac{\sinh 2R}{\cosh 2R - \cos 2k} \operatorname{sech} \xi_{i+1} \times \left(\cos \eta_i \, (\cosh 2R + \sinh 2R \tanh \xi_i) - \cos(2k - \eta_i)\right),$$

$$S_i^y(t) = \frac{\sinh 2R}{\cosh 2R - \cos 2k} \operatorname{sech} \xi_{i+1} \times \left(-\sin \eta_i \, (\cosh 2R + \sinh 2R \tanh \xi_i) - \sin(2k - \eta_i)\right),$$

$$S_i^z(t) = 1 - \frac{\sinh^2 2R}{\cosh 2R - \cos 2k} \operatorname{sech} \xi_i \operatorname{sech} \xi_{i+1}$$



$$\xi_n(t) = 2R\left(n - x_0 - \frac{1}{2}\right) - 2t \sinh 2R \sin 2k,$$

$$\eta_n(t) = -2k\left(n - x_0 - \frac{1}{2}\right) + \eta_0 + 2t\,(1 - \cosh 2R \cos 2k)$$

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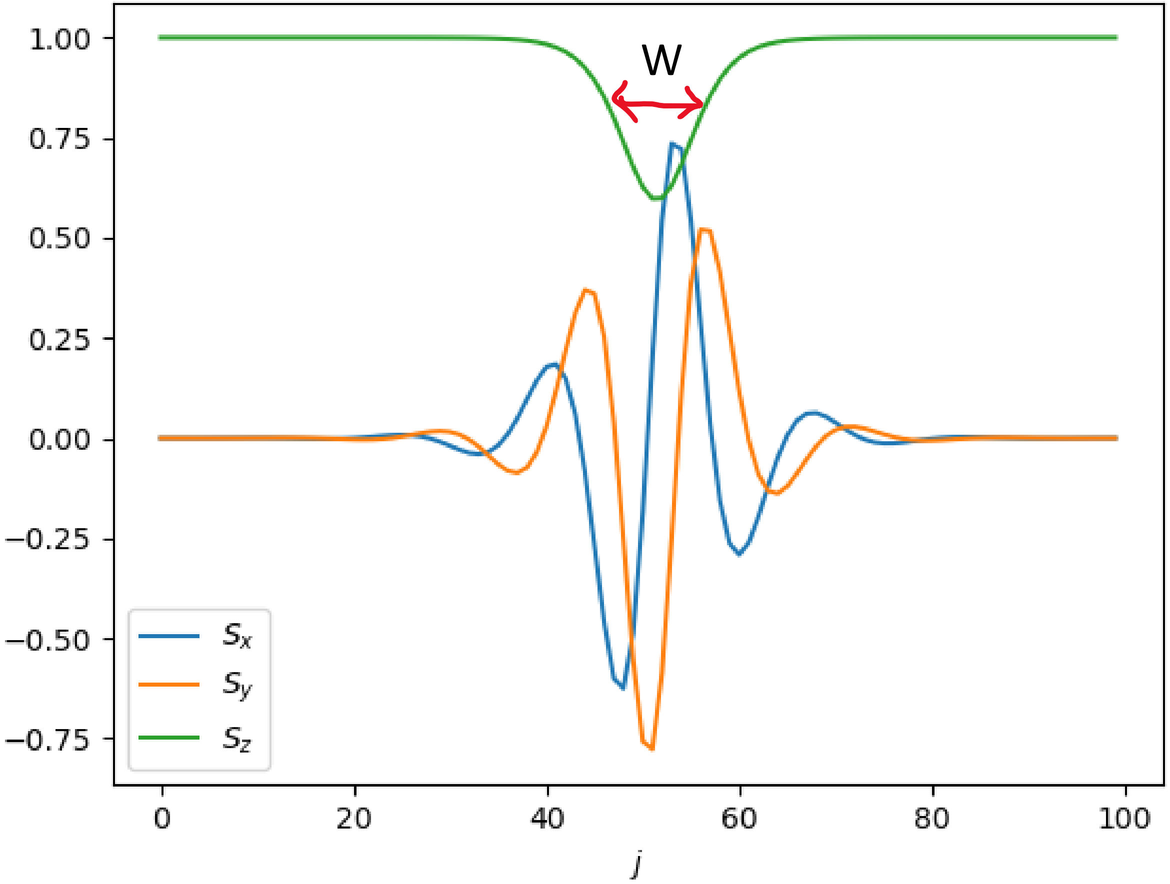

Properties

- Fully determined by two parameters
- R and k

$$W_{1/2}(R) = \frac{\operatorname{arccosh}(2 + \cosh 2R)}{4R}$$

$$E(R, k) = 8R$$

$$M(R, k) = \frac{2 \sinh 2R}{\cosh 2R - \cos 2k}.$$

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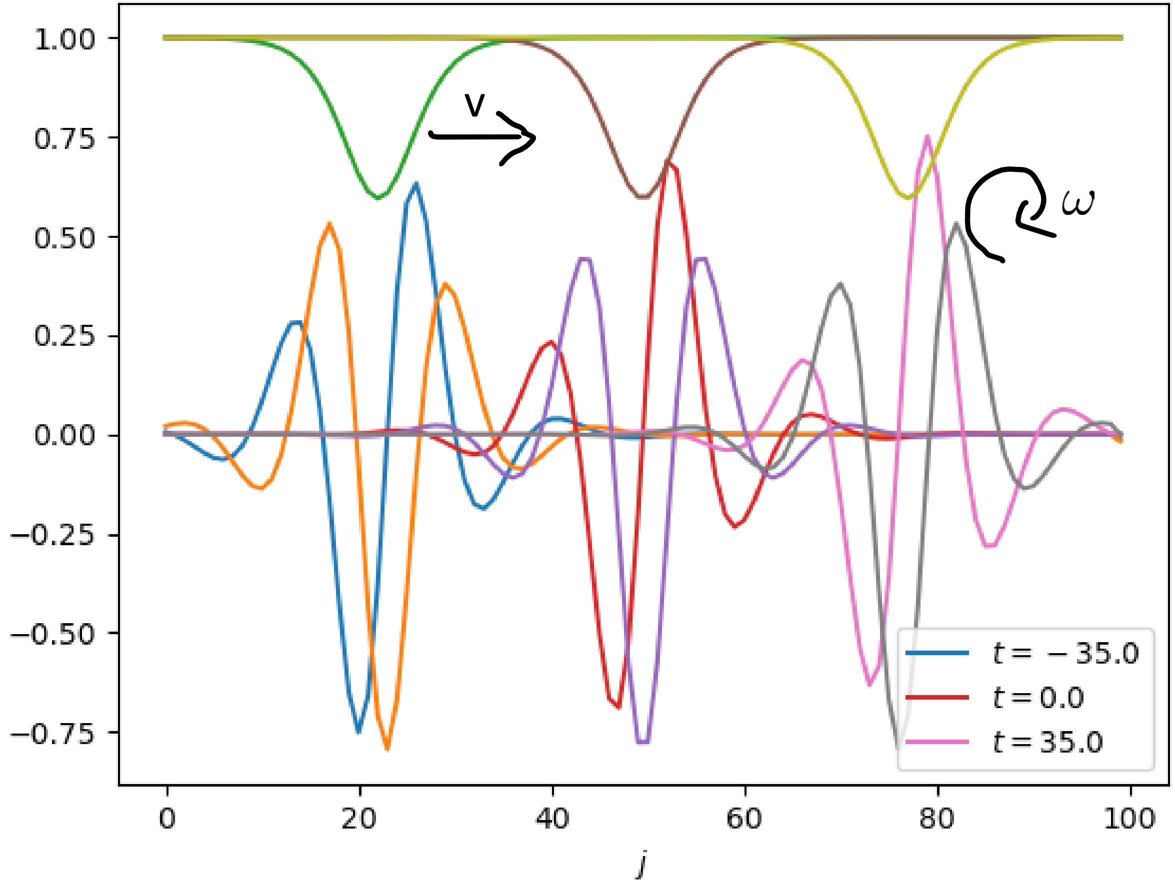

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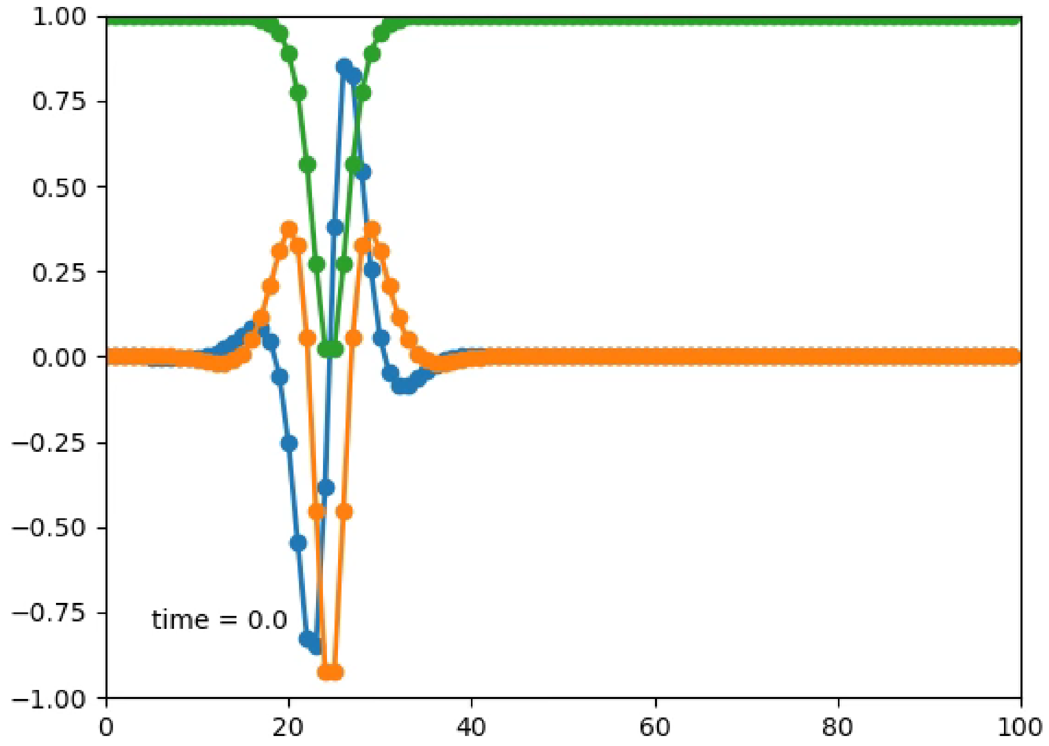

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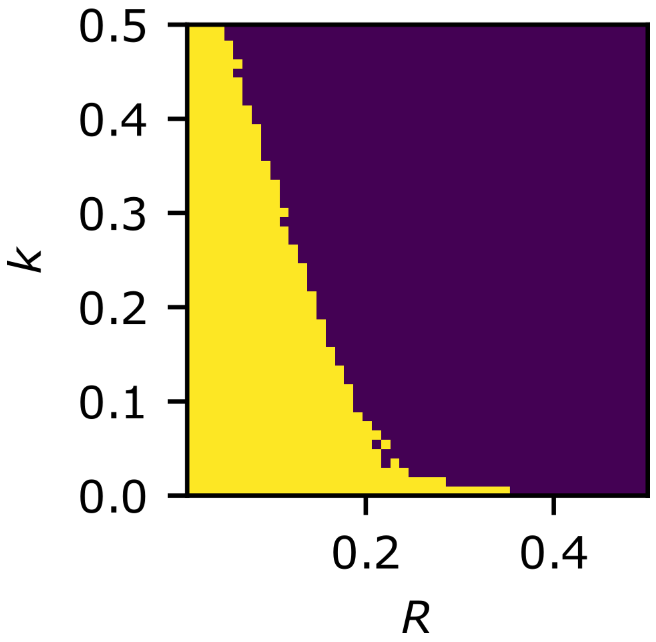
Existence Diagram of Solitons in Heisenberg

1-Soliton Solutions

- Classified by two parameters R and k
- Determined via explicit construction (stationary solitons)
- Adiabatic connection to Ishimori (moving solitons)



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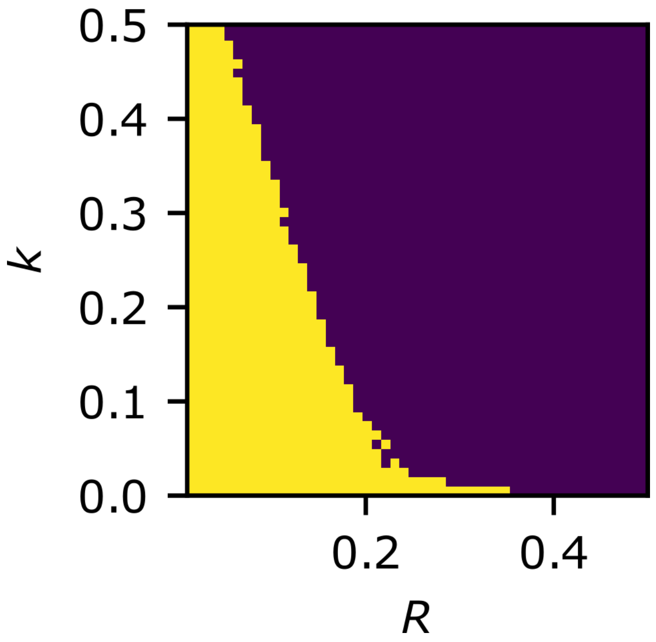

- Exist for all widths R at 0 velocity (k=0)
- At large velocities, only wide solitons are stable

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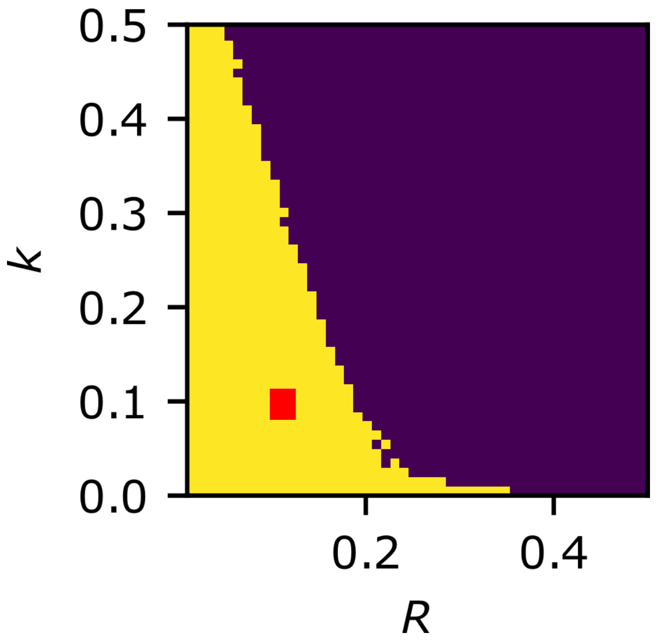

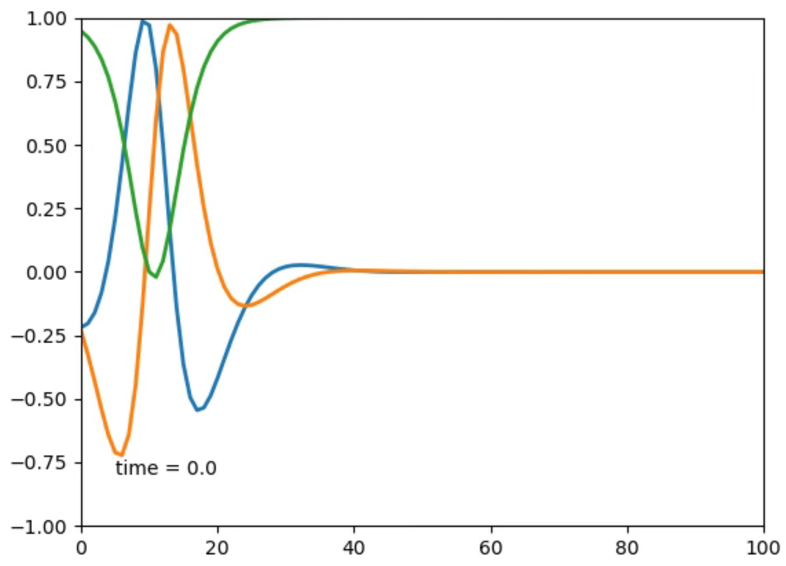

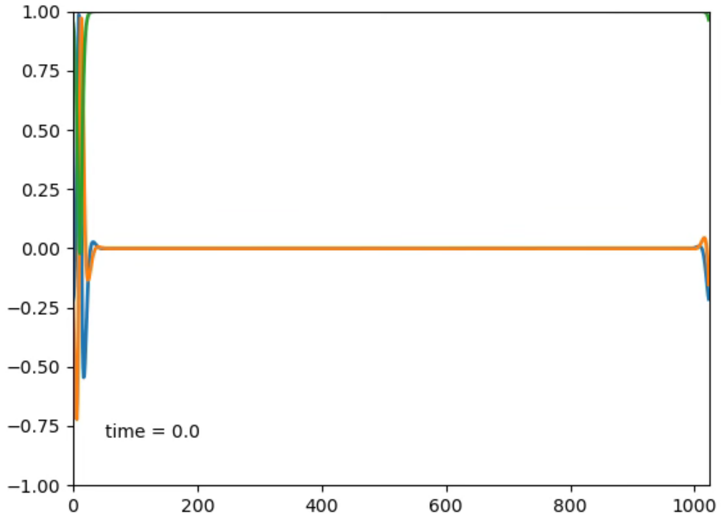

Construction for stationary Solitons

Ansatz

- Stationary z-components
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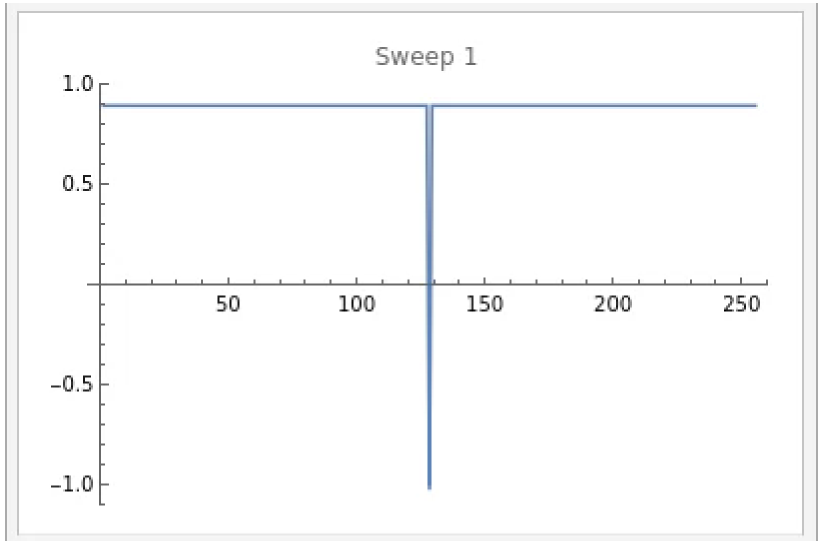

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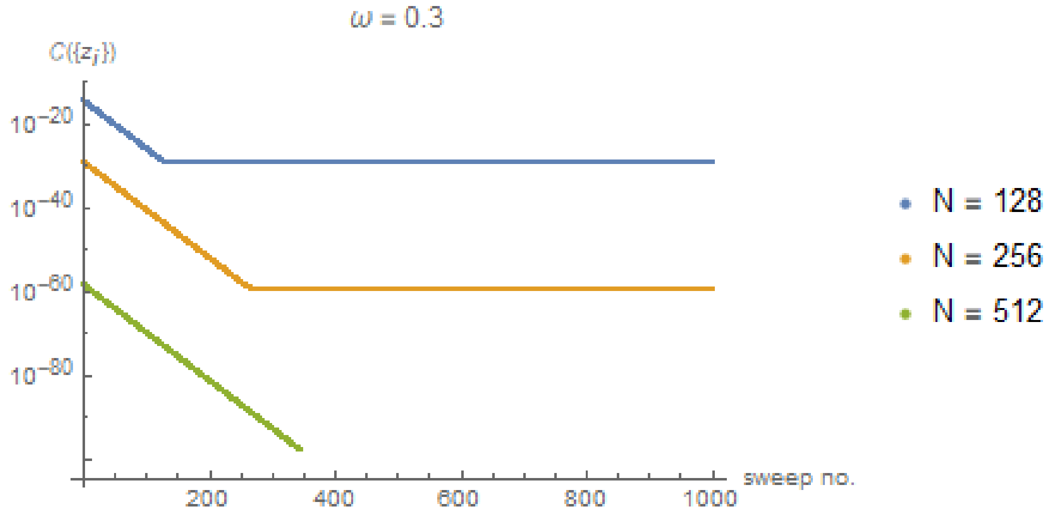

Adiabatic Connection between Heisenberg and Ishimori

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- Smoothly interpolates between HB and Ishimori
- O(3) symmetric for all γ

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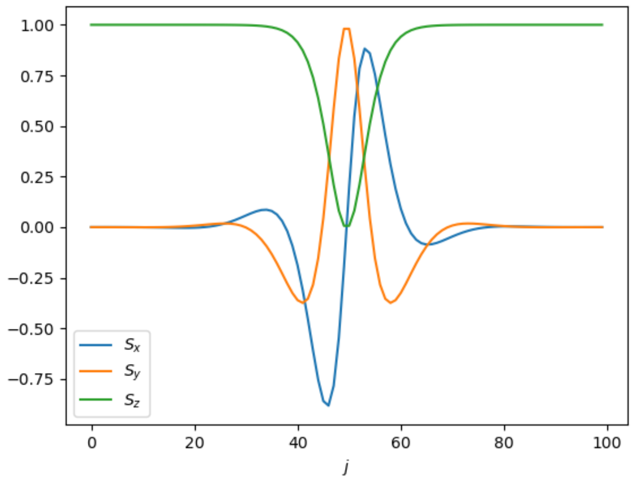

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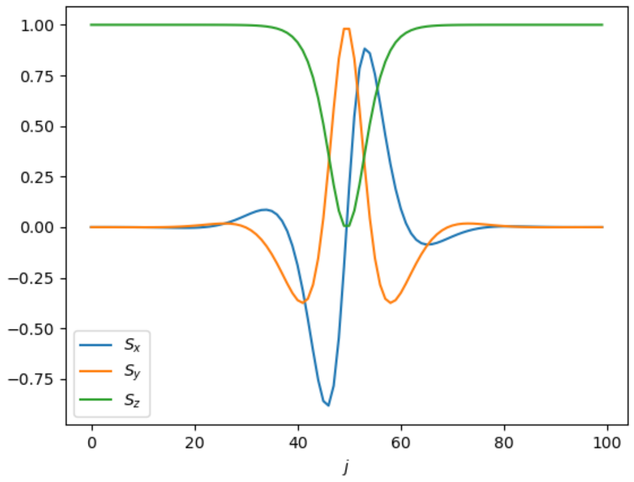
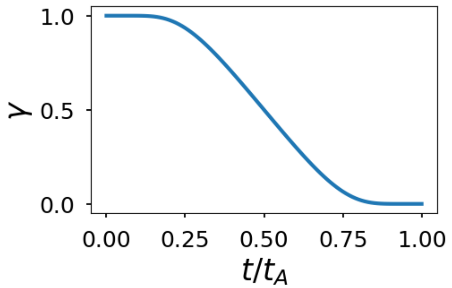


Adiabatic Time-Evolution



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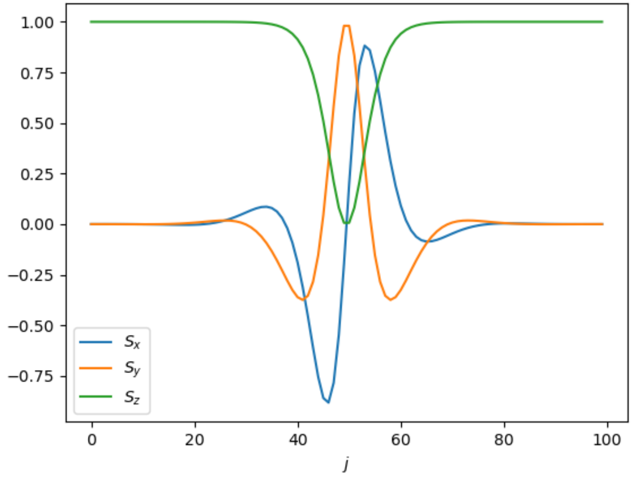

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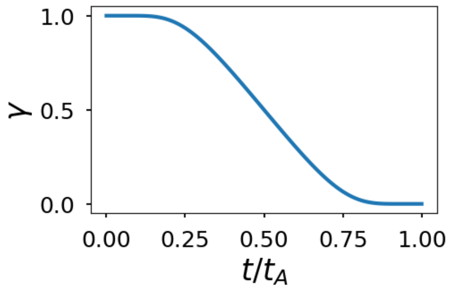

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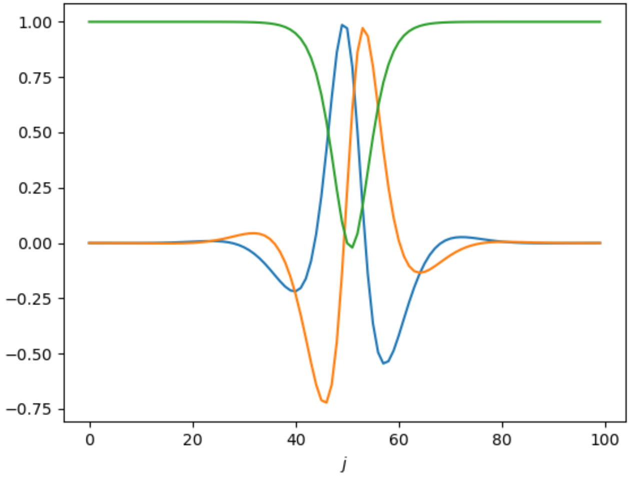
Heisenberg Soliton



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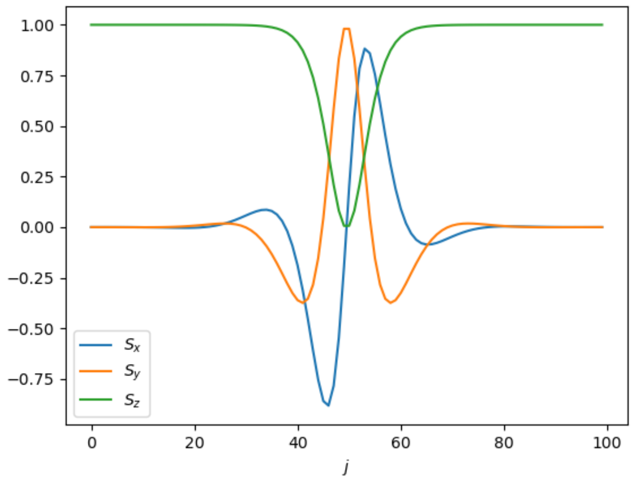

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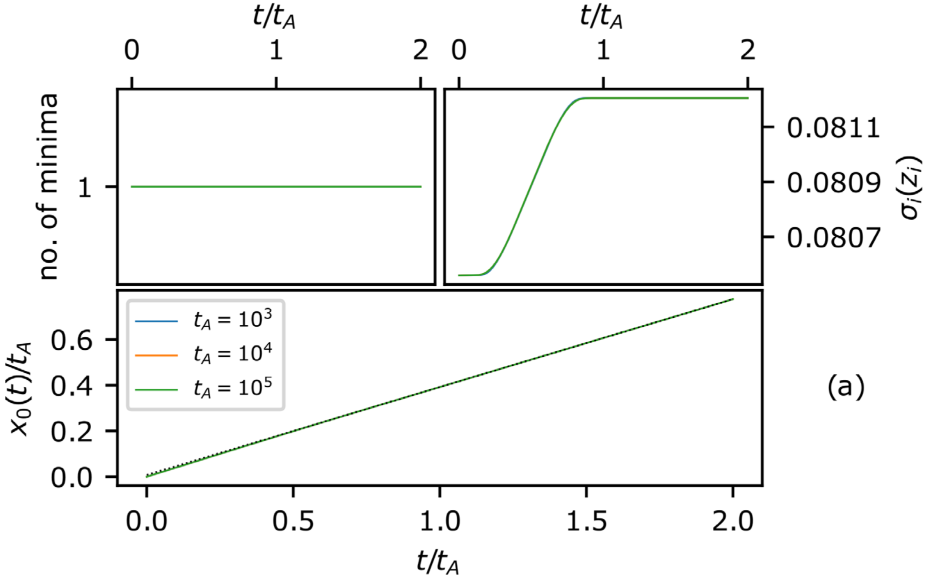

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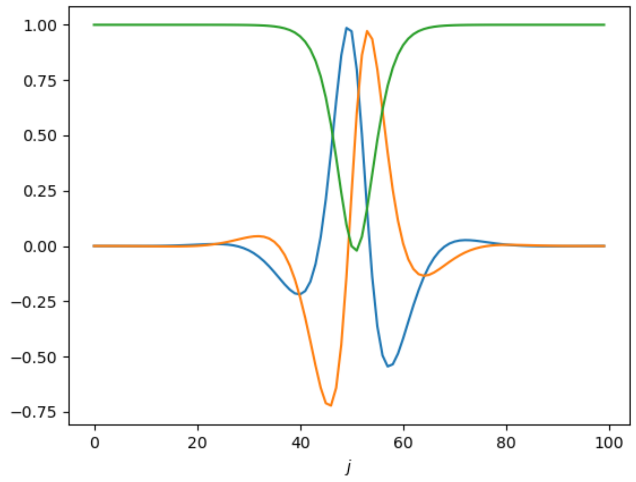

Failure of Adiabatic Connection (narrow solitons)

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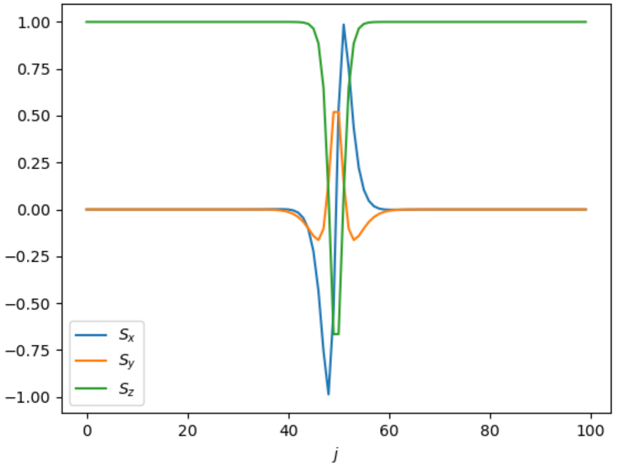
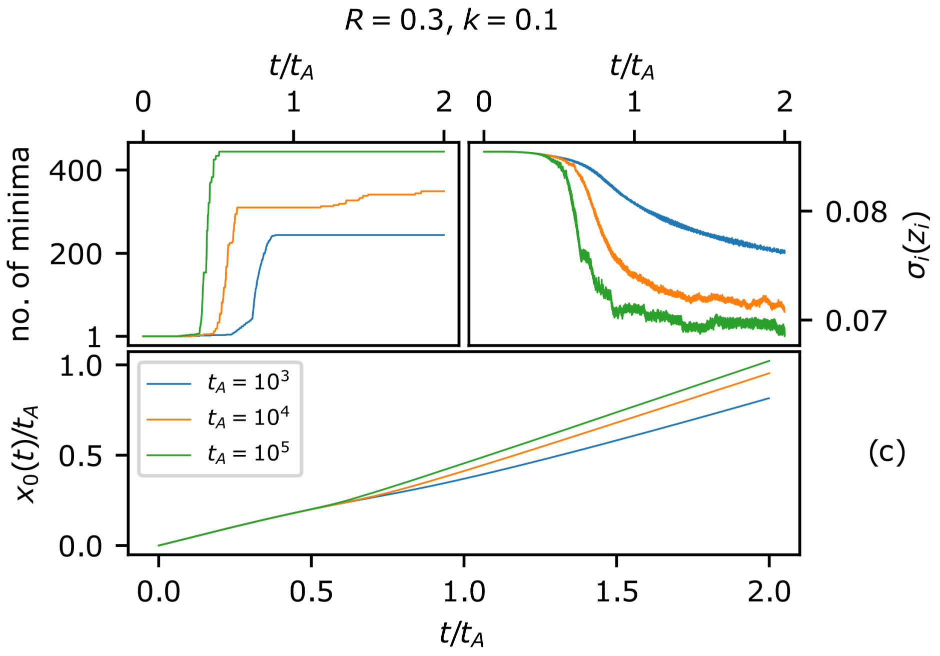
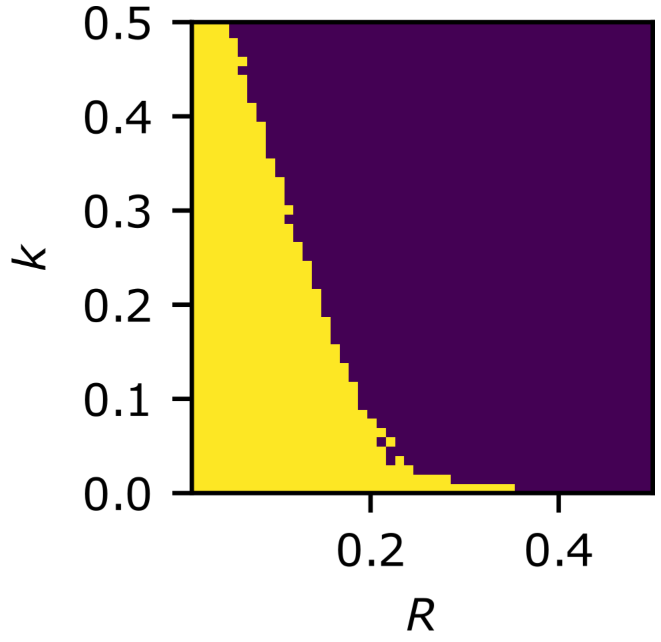




Properties of 1 - Soliton Solutions



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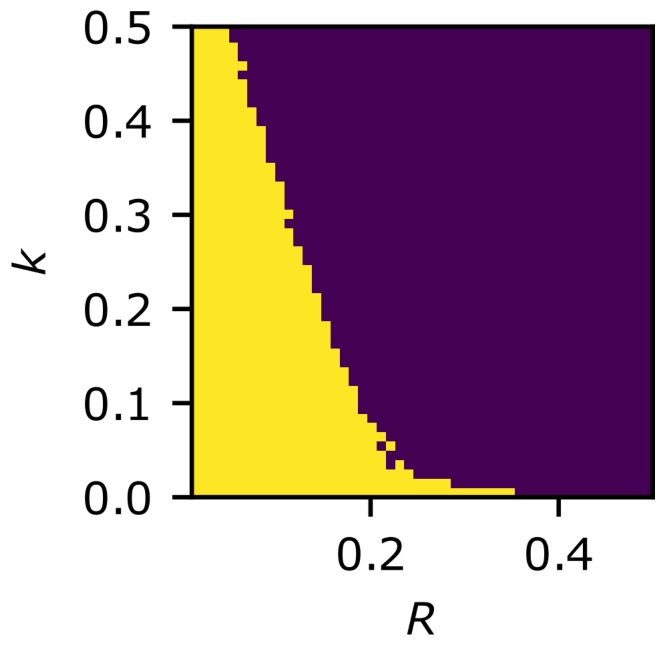

Properties

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  - Velocity v
  - Internal rotation frequency ω
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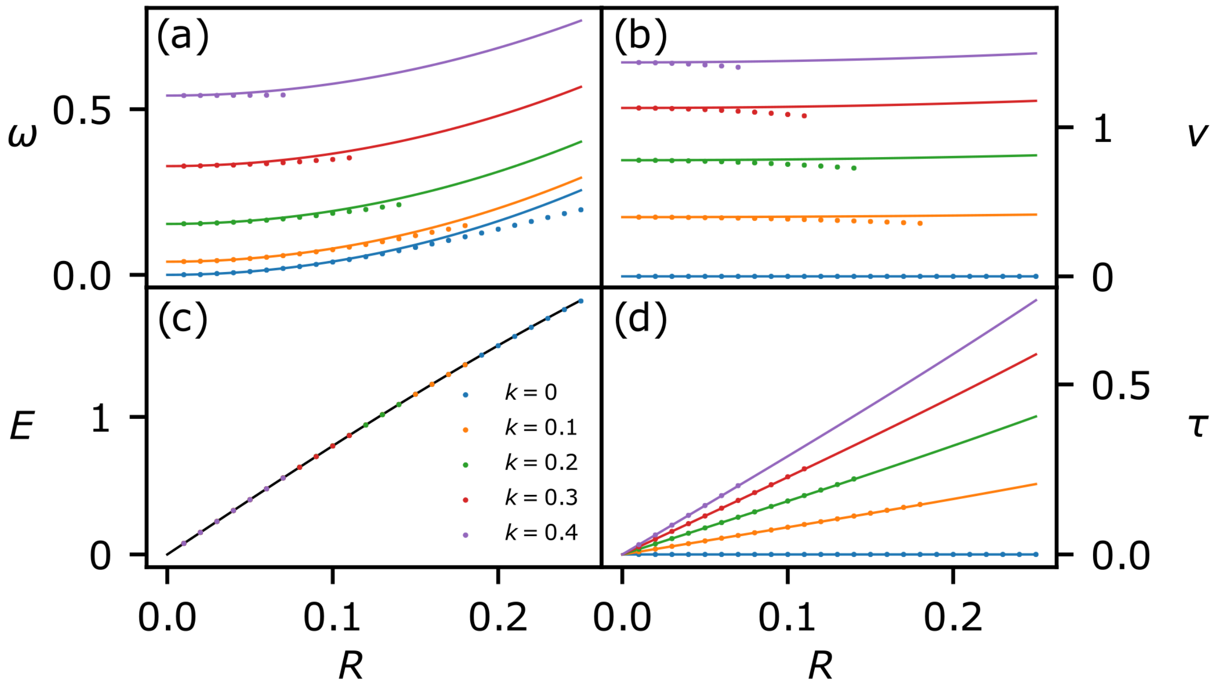

Properties

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- Only slightly renormalized in HB chain

Scattering Properties Ishimori

Integrable Scattering

- Ishimori integrable
- Solitons scatter ballistically
- determined by 2-soliton phase-shifts

2 soliton Phase-Shift [1]

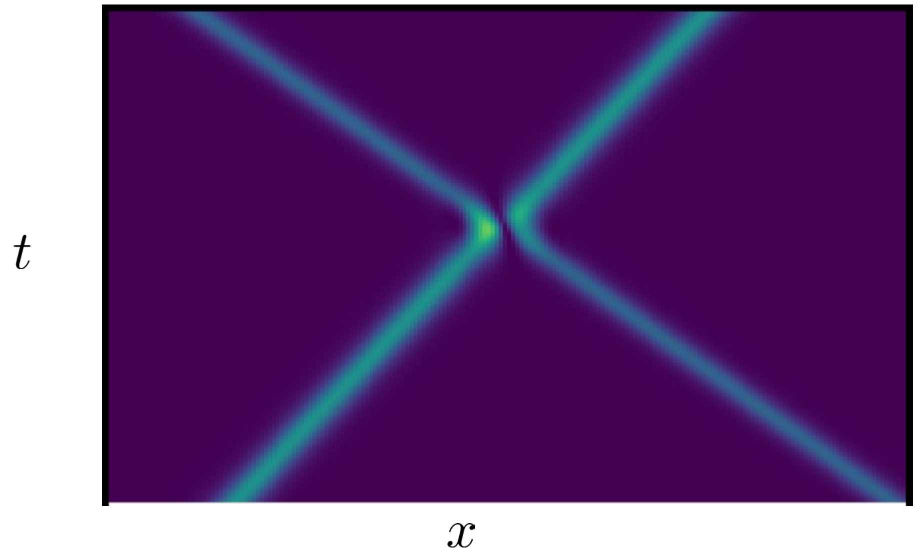
$$\Delta(R,k;R',k') = \mathrm{sgn}(v(R,k) - v(R',k'))$$
$$\times \frac{1}{2R} \log\left[\frac{\cosh(2(R+R')) - \cos(2(k-k'))}{\cosh(2(R-R')) - \cos(2(k-k'))}\right]$$

[1] N.Theodorakopoulos, Physics Letters A, Volume 130

2 – Soliton Scattering

Scattering

- Scatter (almost) ballistically



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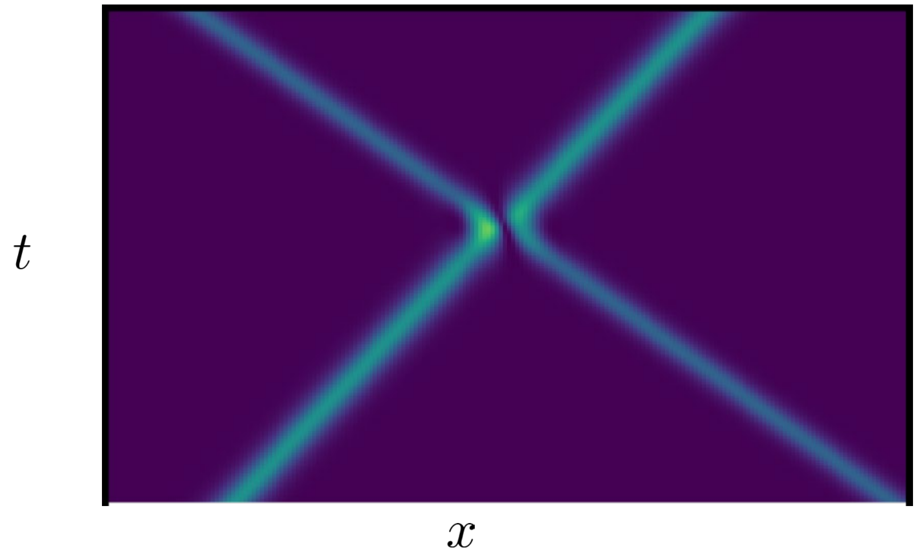

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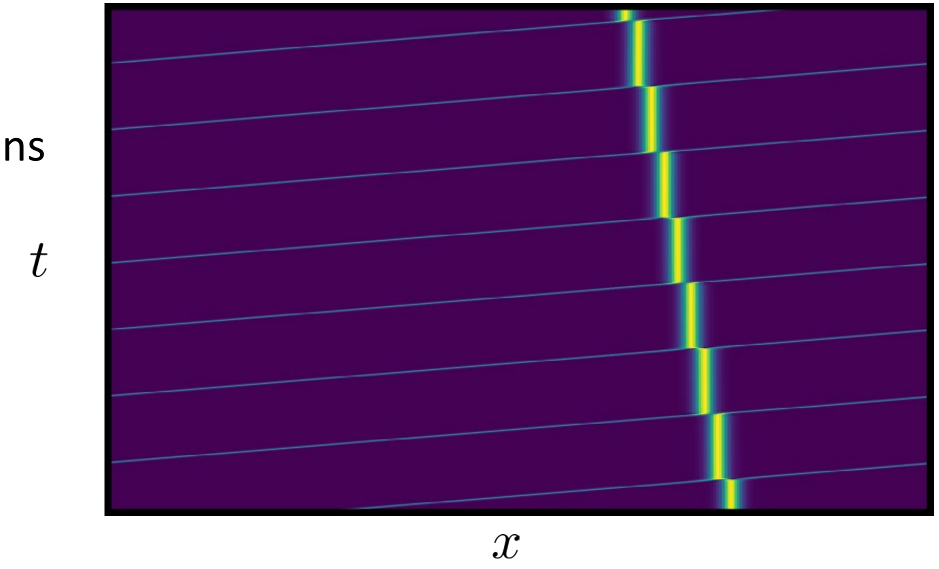

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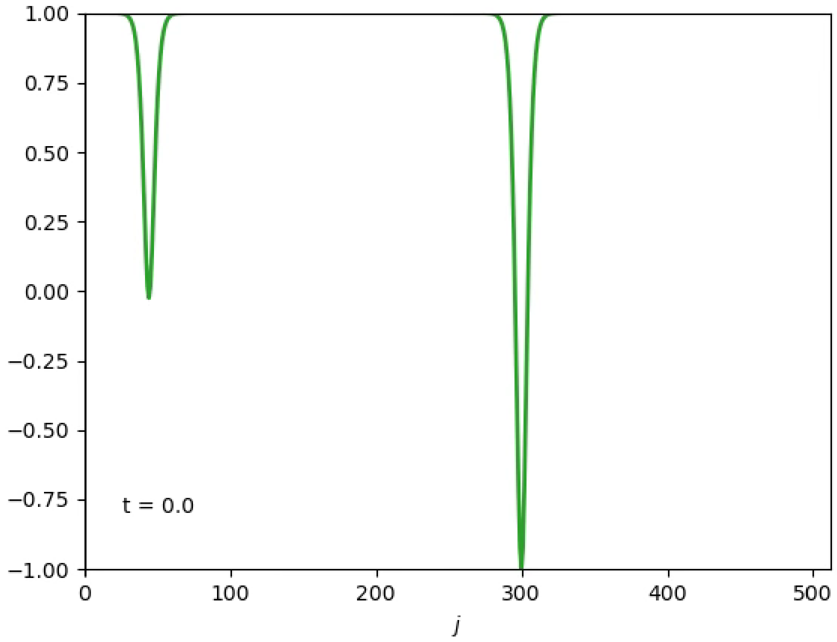

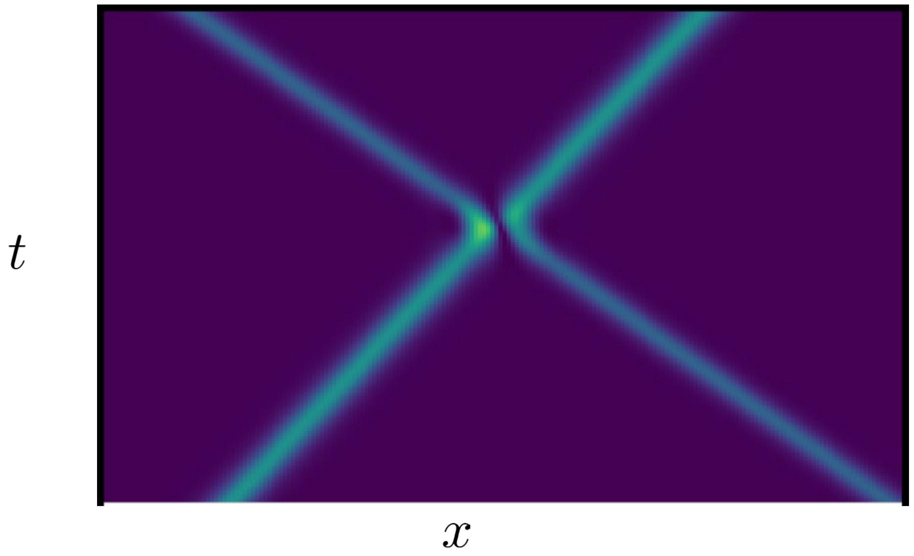

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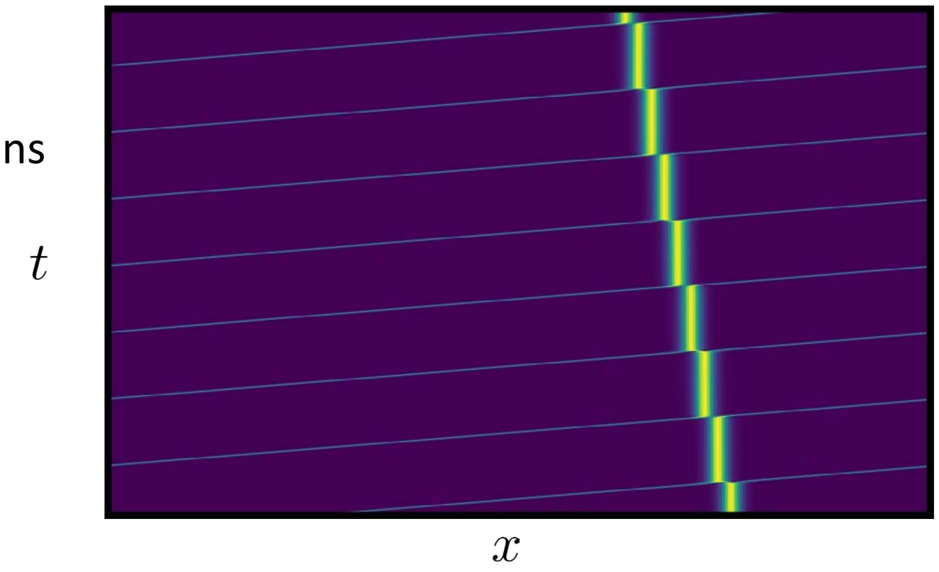
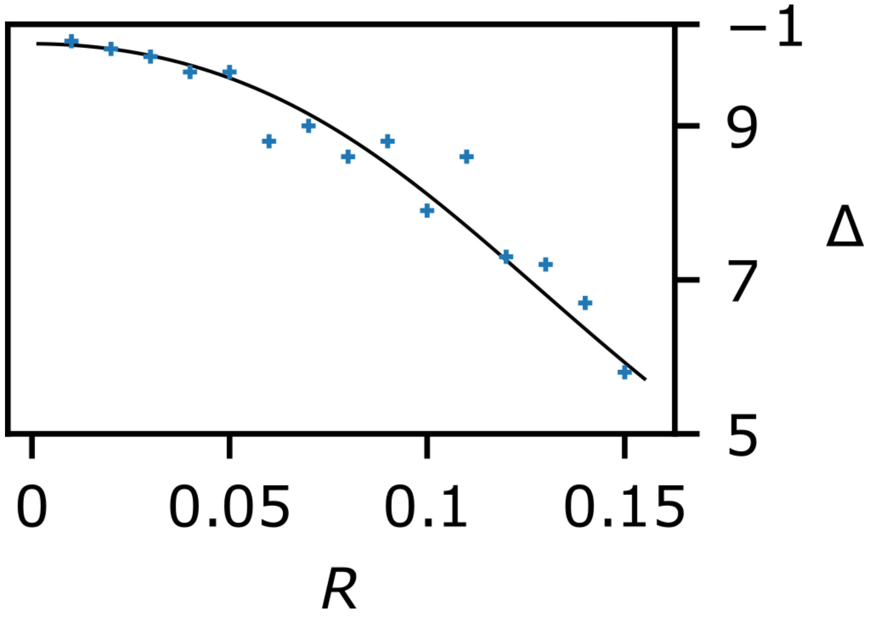
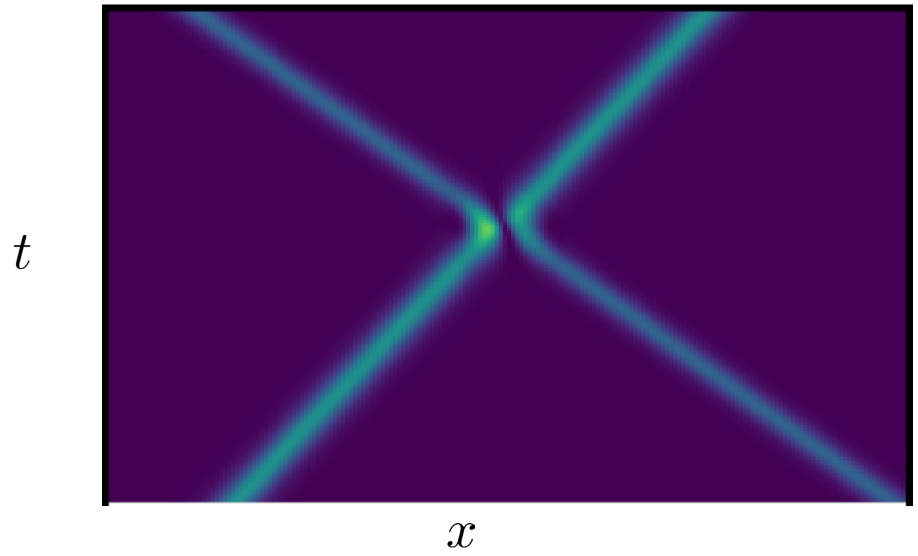


2 – Soliton Scattering

Scattering

- Scatter (almost) ballistically
- can be understood as a phase-shift





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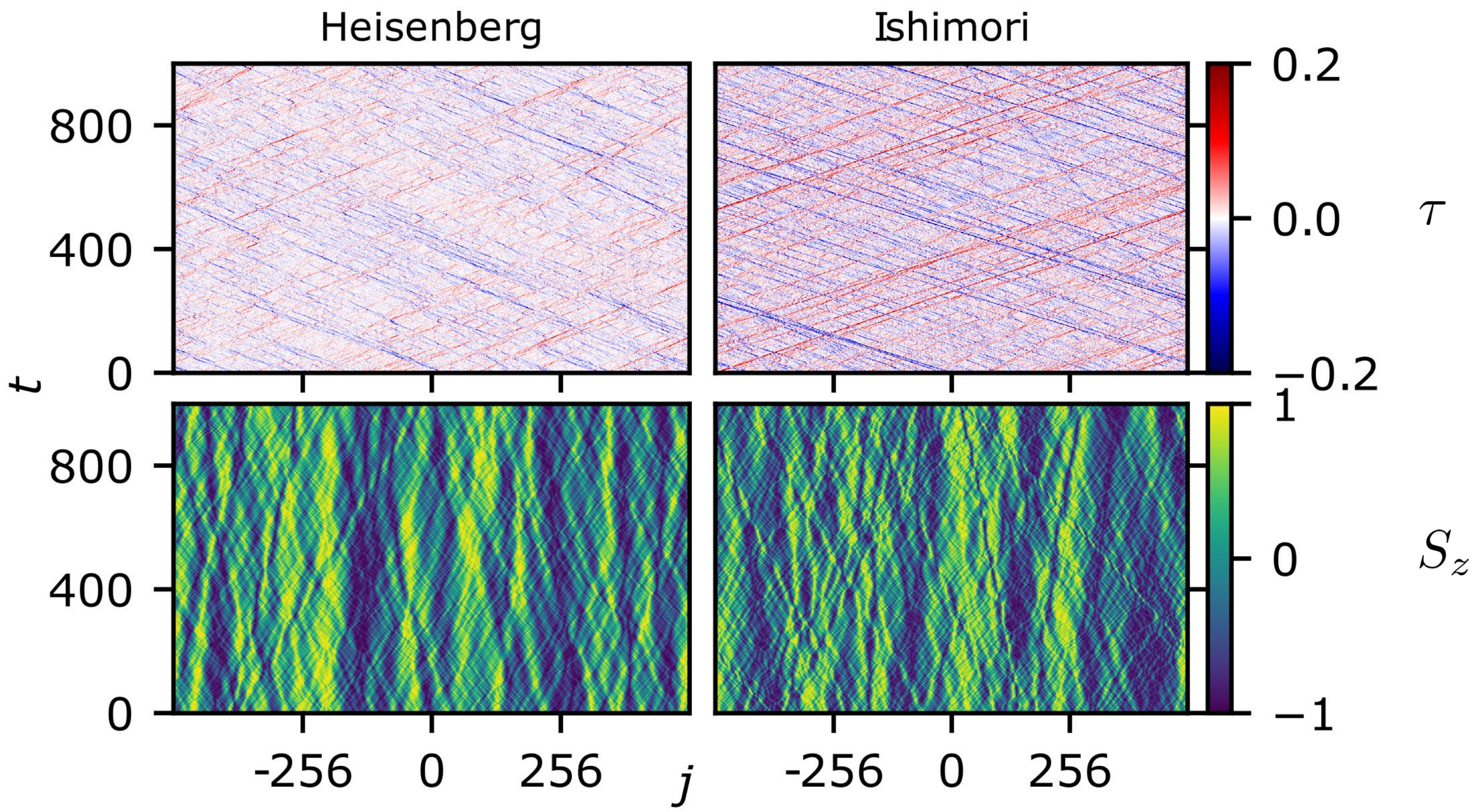
[1] N.Theodorakopoulos, Physics Letters A, Volume 130

Solitons in thermal states



Observations

- Appearance of ballistic trajectories
- long life-times/scattering times

- Torsion fully ballistic
- Magnetisation screened

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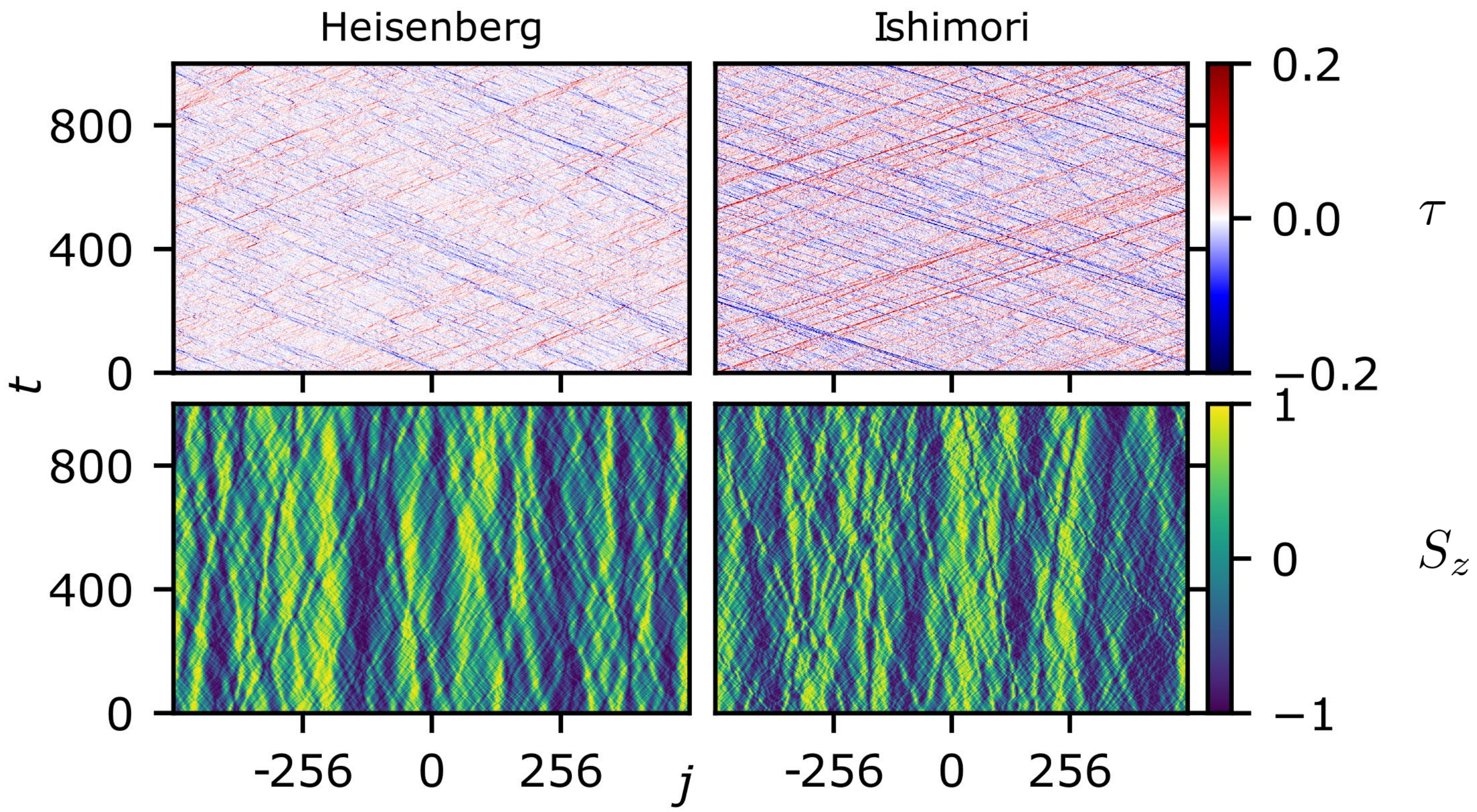

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Solitons in thermal states: Inverse Scattering

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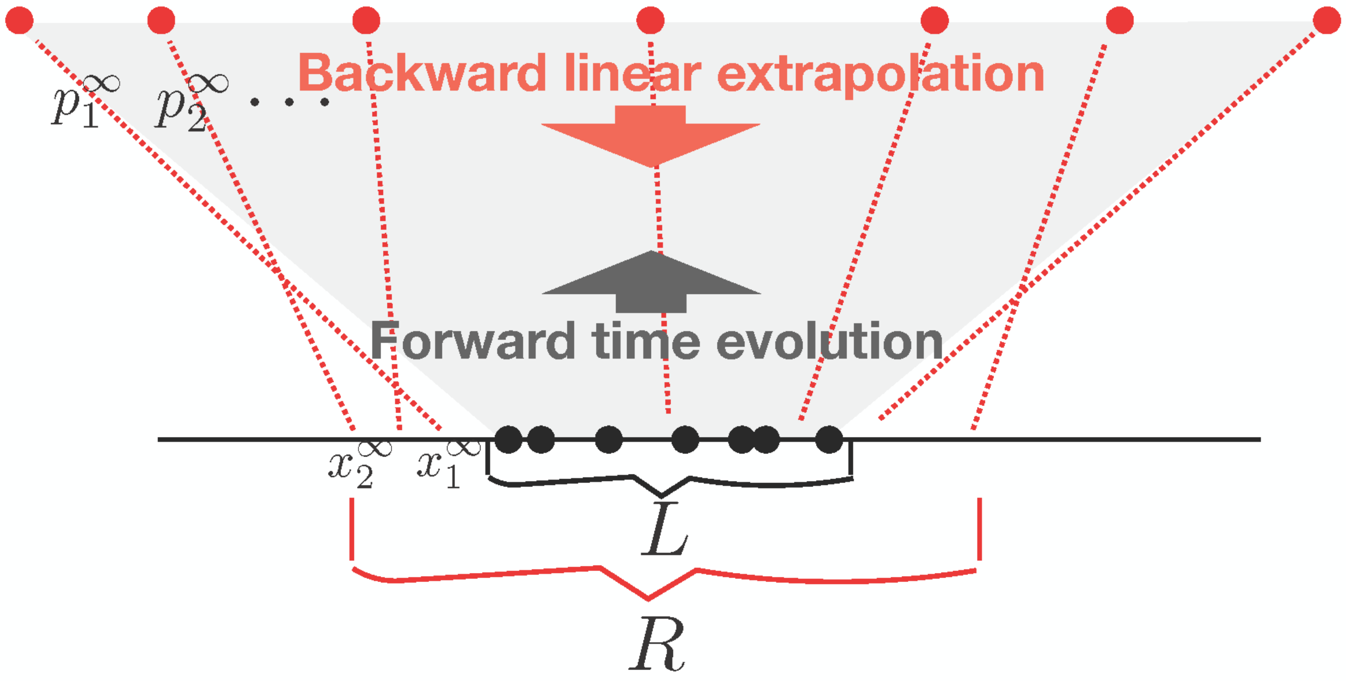

[1] B. Doyon arXiv:1912.08496 (2020)

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$$\mathcal{H} = -2J \sum_i \log\left(\frac{1 + \boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1}}{2}\right)$$

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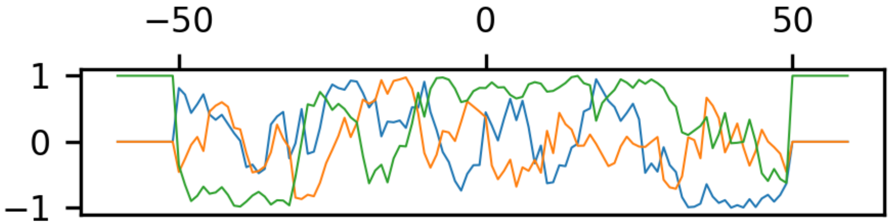

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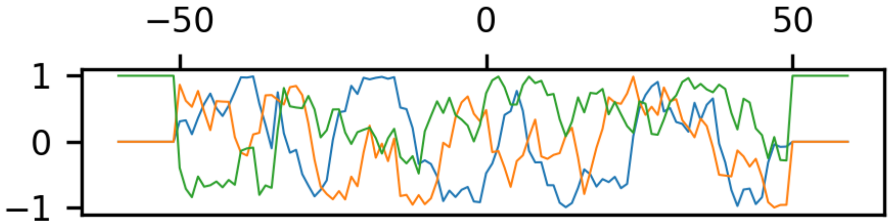

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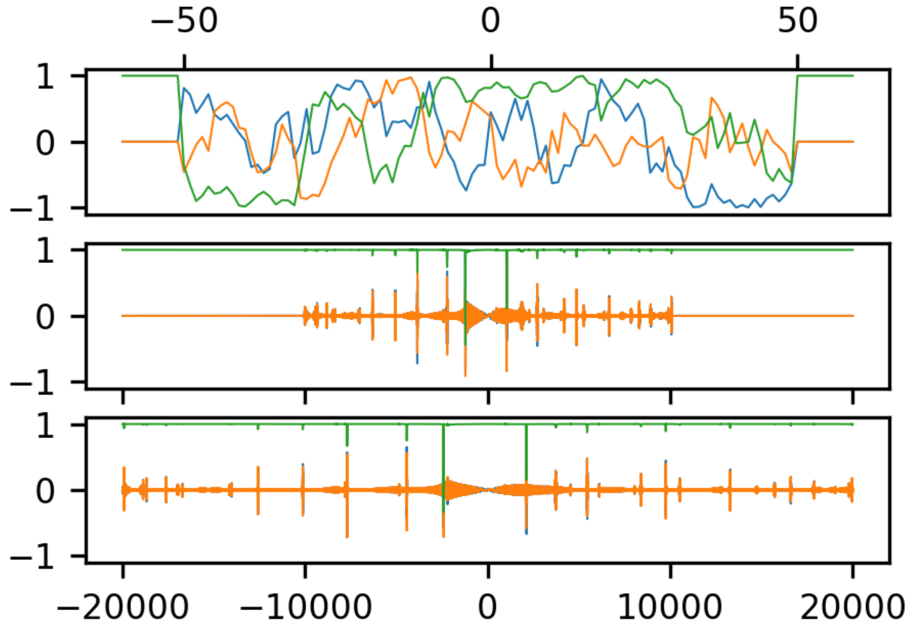
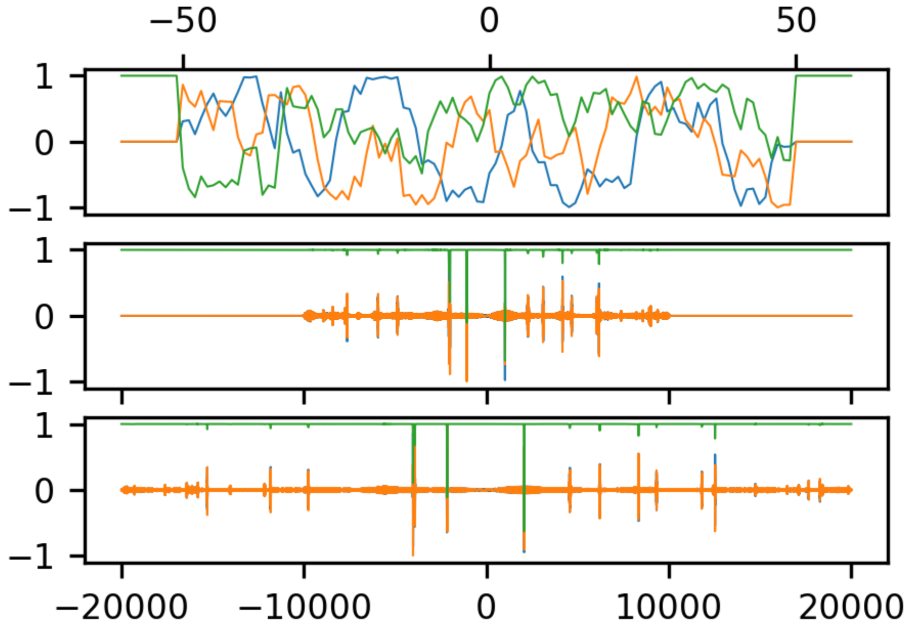


Inverse Scattering

- **Expand a thermal state into vacuum**

$$\mathcal{H} = -J\sum_i (\boldsymbol{S}_i\cdot\boldsymbol{S}_{i+1})$$



Solitons in thermal states: Inverse Scattering

Question

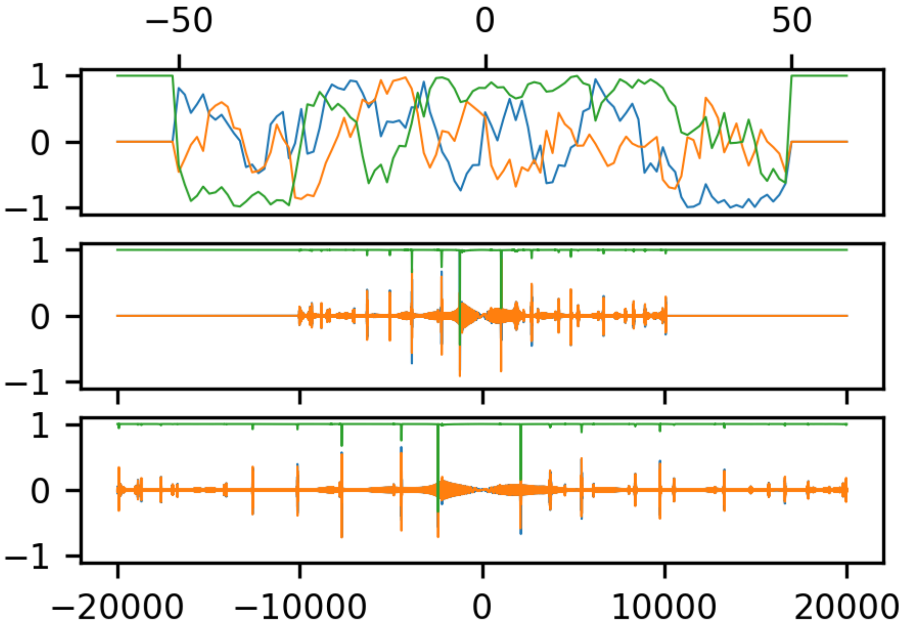
- Ishimori state composed of solitons
- **Is this true for Heisenberg ?**

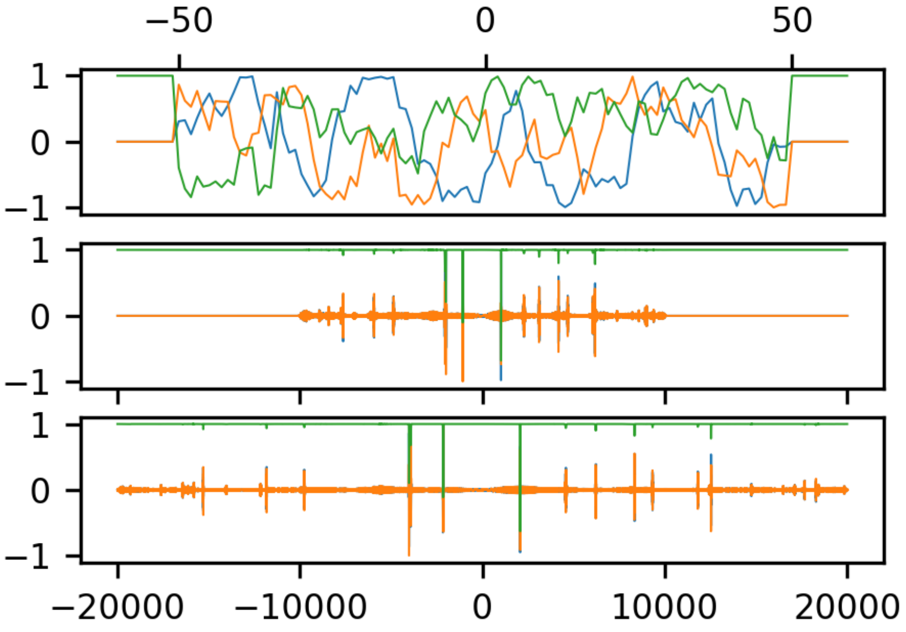
Inverse Scattering

- **Expand a thermal state into vacuum**

$\mathcal{H} = -2J \sum_i$ l… $(\boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1})$

Thermal states contain significant soliton content





Solitons, Screening & KPZ

- Thermal states contain long-lived solitons
- Solitons scatter close to integrable

[1] S.Gopalakrishnan, R. Vasseur , PRL, 127202 (2019)

Solitons, Screening & KPZ

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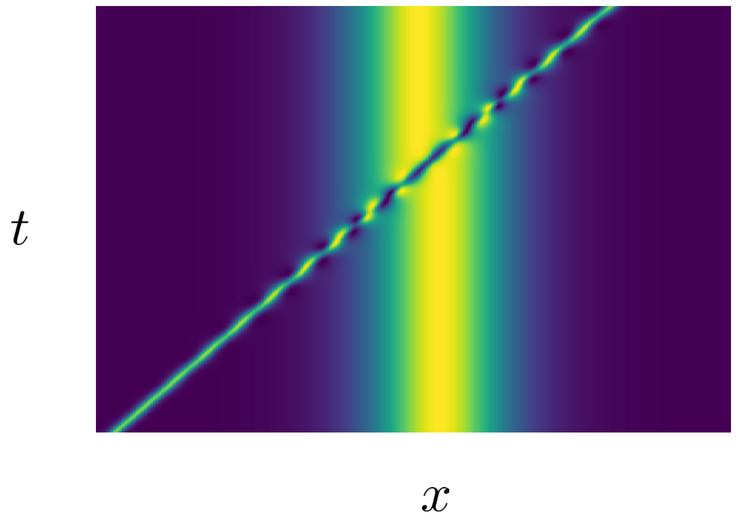

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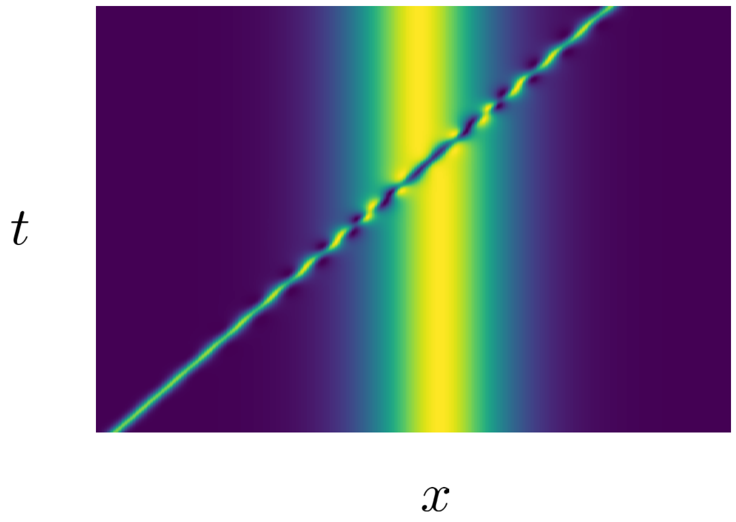
- Mechanism invoked to explain KPZ scaling in XXZ quantum Heisenberg [1]



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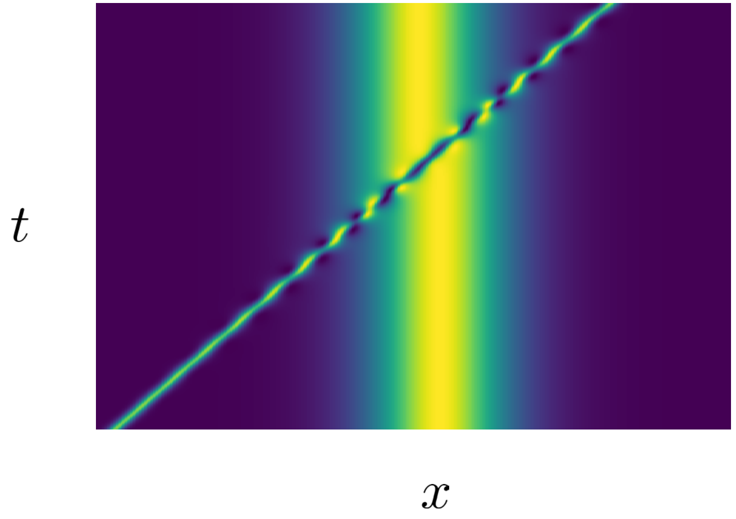
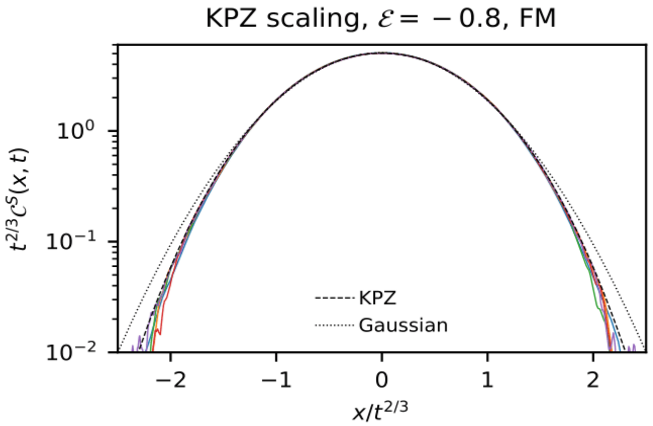
Solitons, Screening & KPZ

- Thermal states contain long-lived solitons
- Solitons scatter close to integrable
- Magnetisation transport is screened by collisions
- Mechanism invoked to explain KPZ scaling in XXZ quantum Heisenberg [1]
- Thus, **potential explanation of observed KPZ scaling**

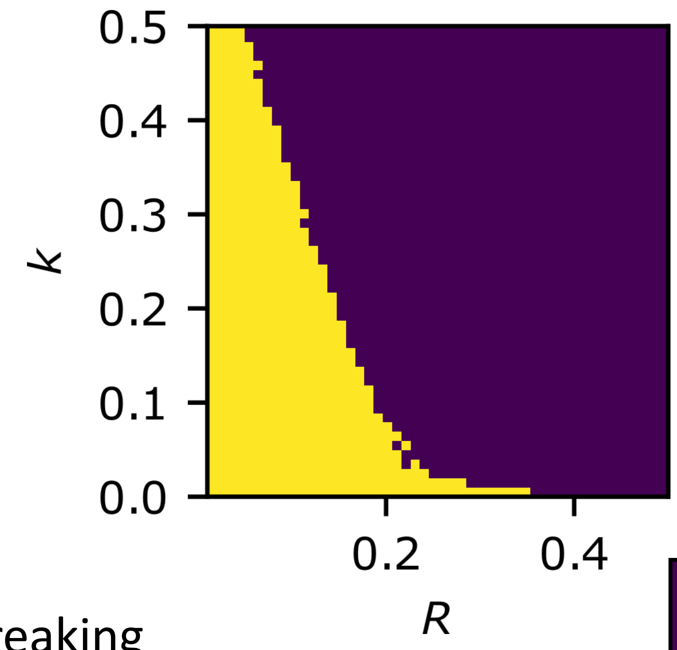


[1] S.Gopalakrishnan, R. Vasseur , PRL, 127202 (2019)

Discussion and Conclusions

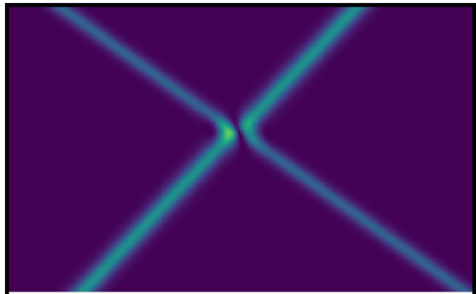


- Classical Heisenberg chain is one of the simplest dynamical models of magnetism
- Hides a rich regime of superdiffusive spin hydrodynamics in the FM
- KPZ scaling at low temperature in the FM



- Adiabatic connection to integrable Ishimori chain
- FM Heisenberg chain hosts exact 1 soliton solutions
- Solitons scatter (almost) integrably
- Thermal states contain/compose of multiple solitons
- Solitons could explain the observed KPZ scaling

- Emergent picture of symmetry-preserving perturbations breaking integrability only very weakly



Credits

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PhD & PostDoc Positions with me at Oklahoma

- Contact me!

References

- **Phys. Rev. B 105, L100403**
- **Arxiv (soon)**

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DFG

mpipks