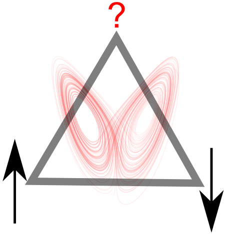
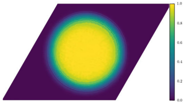
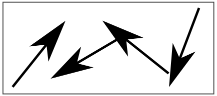
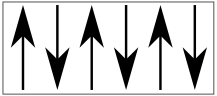
Many-body chaos in classical spin systems

with and without quasi-particles









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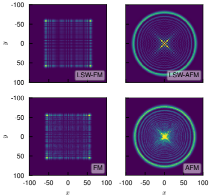


Phys. Rev. Lett. 121, 250602 (2018)

Phys. Rev. B 103, 174302 (2021)

Overview

Basic issue: Chaos in many-body physics

- phenomenology/description
- microscopics $\leftrightarrow$ macroscopics
- chaos $\leftrightarrow$ Phases/Symmetry breaking
- chaos $\leftrightarrow$ classical order/quasi-particles

Recent Developments

- quantum chaos (OTOC's)
- bounds on chaos
- new spin liquids

Nature of chaos in classical many-body physics

- butterfly effect
- Lyapunov exponent
- diffusion

Platform

- Classical $\mathbb{Z}_2$ spin liquid
  - no order/no quasi-particles
- (Anti)Ferromagnets
  - Classical Order/Spin-waves

(Not an introduction to) chaos

Chaos plays many roles

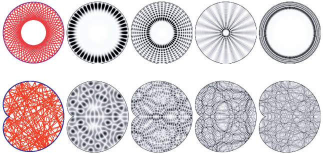
- intrinsically fascinating

Underpins thermodynamics

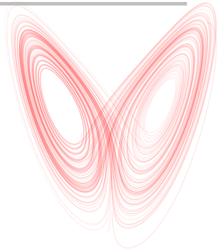
- many-body systems generically chaotic

Connection classical $\leftrightarrow$ QM?

- semiclassics and beyond

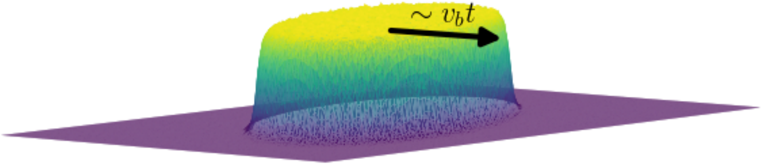


(Classical) Chaos and the butterfly effect



Butterfly effect: spatiotemporal phenomenon

- sensitive dependence on initial conditions
  - butterfly wingbeat $\rightarrow$ 'leads to' catastrophic weather event
  - Lyapunov exponent: perturbation grows $\sim e^{\lambda t}$
- propagates in space
  - weather event takes place far form the butterfly
  - butterfly speed: $x \sim v_b t$

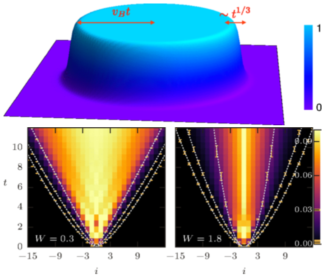


Chaos in QM (OTOC's)

OTOC (out-of-time-ordered correlator) (Larkin '69)

- $\left\langle \left[\hat{W}(t), \hat{V}(0)\right]^2 \right\rangle$
- shows butterfly effect
- Quantum chaos bound $\lambda \lesssim T$ (Maldacena '15)

- experimental measurements (Bollinger/Rey '17)
- random circuits (Nahum '18, Keyserlingk '18)
- random Heisenberg chain (Luitz '17)
- Luttinger liquids (Dora '17)
- kicked rotor (Galitski '17)
- quantum bound (Maldacena '15)
- black holes (Shenker '14)
- SYK/holography ('93,'15)
- large-N theories
- . . .

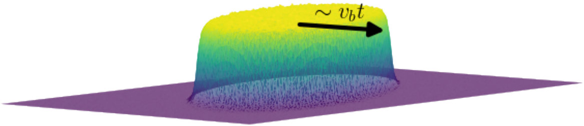


Decorrelator: classical limit of OTOC

$$2D(\mathbf{x}, t) = \left\langle [\delta \mathbf{S}_{\mathbf{x}}(t)]^2 \right\rangle \approx \epsilon^2 \left\langle \{\mathbf{S}_{\mathbf{x}}(t), \mathbf{n} \cdot \mathbf{S}_0(0)\}^2 \right\rangle$$ [1]

Intuitive: distance of perturbed trajectories

- $\left|\tilde{\mathbf{S}} - \mathbf{S}\right|^2 = |\delta \mathbf{S}|^2$



Semiclassical version of OTOC

- $[\cdots,\cdots] \to \{\cdots,\cdots\}$
- also equations of motion: natural semiclassical limit

[1] Das et al., PRL 121.024101 (2018)

Classical Decorrelators

$$\boxed{2D(\mathbf{x}, t) = \left\langle [\delta \mathbf{S_x}(t)]^2 \right\rangle \approx \epsilon^2 \left\langle \{\mathbf{S_x}(t), \mathbf{n} \cdot \mathbf{S}_0(0)\}^2 \right\rangle}$$

- Light-cone spreading of perturbations and the butterfly effect in a classical spin chain : Das et al. (2018)
- Temperature dependence of butterfly effect in a classical many-body system : TB et al (2018)
- Many-body Chaos in a Thermalised Fluid: Kumar et al (2019)
- Many-body chaos near a thermal phase transition: Schuckert et al (2019)
- Many-body chaos and anomalous diffusion across thermal phase transitions in two dimensions: Ruidas et al (2020)
- Butterfly Effect and Spatial Structure of Information Spreading in a Chaotic Cellular Automaton: Liu et al (2021)
- Classical many-body chaos with and without quasiparticles: TB et al (2021)

Phenomenology in a classical many-body system

Q: Classical Chaos

- Chaos for $T \in [0, \infty[$?
- low T scaling of chaos?
- (classical) OTOC's?

Q: Fundamental

- Relations between microscopic chaos and macroscopic dynamics?
- Effect of (thermal) phase transitions?
- Effects of Symmetry-Breaking
- Relevance of quasi-particles

classical spin liquid

- non-trivial dynamics down to $T \to 0$
- single regime – no ordering

Classical FM/AFM

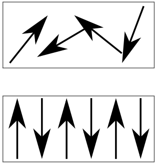
- thermal phase transition
- quasi-particles/spin-waves

Magnetic Order & Frustration & Classical Spin-liquids

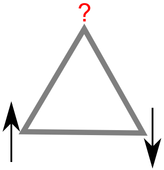
Frustration $\longrightarrow$

ferromagnetism anti-ferromagnetism ferrimagnetism metamagnetism ⋮	geometrical frustration

- Frustration / Competing interactions
- Destabilisation of conventional order



- Thermal fluctuations destroy order
  - Paramagnet $\leftrightarrow$ FM/AFM
- Competing interactions suppress order
  - strongly interacting for $T \ll J$
    - cooperative paramagnet

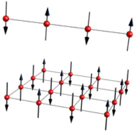


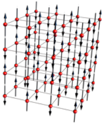
Classical Heisenberg magnets & Spin-liquids

$$\mathcal{H} = J \sum_{ij} \mathbf{S}_i \cdot \mathbf{S}_j$$

Cubic lattices

- Spin-Chains (1D)
- square (2D)
- cubic (3D)

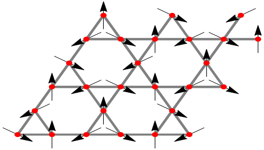




- Ferromagnets $J < 0$
- Anti-Ferromagnets $J > 0$

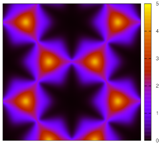
Lattices

- Kagome (2D)
- Pyrochlore (3D)



Many types

- U(1)/Coulomb
- "$\mathbb{Z}_2$" (Rehn '17)
- "jammed" SL (Bilitewski '17)



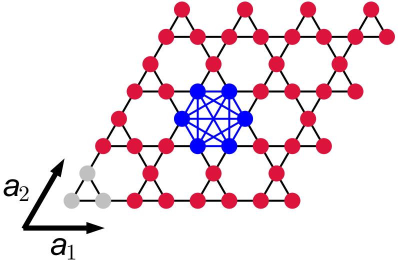
Part I: Chaos in a spin liquid

Kagome lattice: corner-sharing hexagons

Model

- Heisenberg O(3) spins
- all spins in a hexagon ⬡ interact

$$H = J \sum_{\mathbf{x},\mathbf{x}' \in \hexagon} \mathbf{S}_{\mathbf{x}} \cdot \mathbf{S}_{\mathbf{x}'} = \frac{J}{2} \sum_{\alpha} (\mathbf{L}_\alpha)^2 + \mathrm{c}$$



Classical "$\mathbb{Z}_2$" spin liquid

- no order as $T \to 0$
- all correlations short-ranged

Semiclassical Dynamics

$$\frac{d\mathbf{S}_{\mathbf{x}}(t)}{dt} = -\mathbf{S}_{\mathbf{x}}(t) \times \sum_{\mathbf{x}'} J_{\mathbf{x}\mathbf{x}'} \mathbf{S}_{\mathbf{x}'}(t)$$

- Initial states sampled via Monte-Carlo

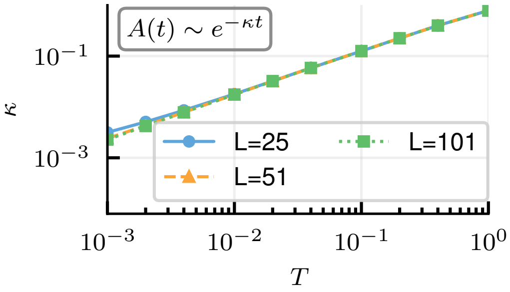
Thermodynamics

$$\boxed{A(t) = \sum_i \langle \mathbf{S}_i(t)\mathbf{S}(0)\rangle}$$

$$\boxed{\mathcal{S}(\mathbf{q},\omega) = \sum_{ij} \int dt\, e^{i(\omega t+\mathbf{q}(\mathbf{r}_i-\mathbf{r}_j))} \left\langle \mathbf{S}_i(t)\cdot\mathbf{S}_j(0)\right\rangle}$$

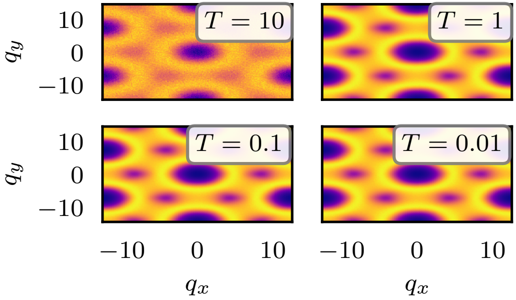
Relaxational dynamics

$A(t) \sim e^{-\kappa t}$



No order

spin diffusion $\mathcal{S}(\mathbf{q},\omega) \sim 1/[(Dq^2)^2 + \omega^2]$



Protocol: exponential sensitivity and chaos

$$\boxed{2D(\mathbf{x},t) = \left\langle [\delta \mathbf{S}_{\mathbf{x}}(t)]^2 \right\rangle \approx \epsilon^2 \left\langle \{\mathbf{S}_{\mathbf{x}}(t), \mathbf{n}\cdot\mathbf{S}_0(0)\}^2 \right\rangle}$$

Perturb initial state $\{\mathbf{S}_0\}$

- $\delta \mathbf{S}_0(t=0) = \epsilon(\mathbf{n}\times\mathbf{S}_0)$

Evolve perturbed state

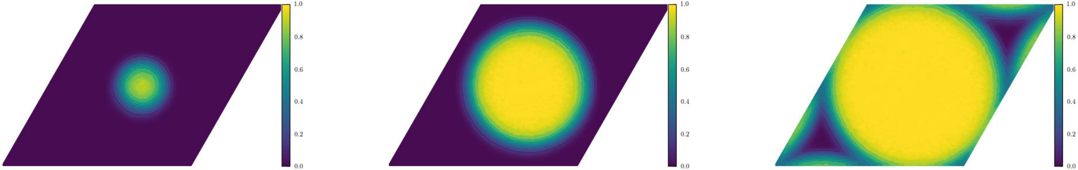
- $\tilde{\mathbf{S}}(t) = \mathbf{S}(t) + \delta\mathbf{S}(t)$

Measure distance via 'decorrelator'

- $D(t) = \left|\tilde{\mathbf{S}}(t) - \mathbf{S}(t)\right|^2$

Extract quantities of interest

Time evolution of decorrelator



Ballistic propagation

- despite diffusive spin density

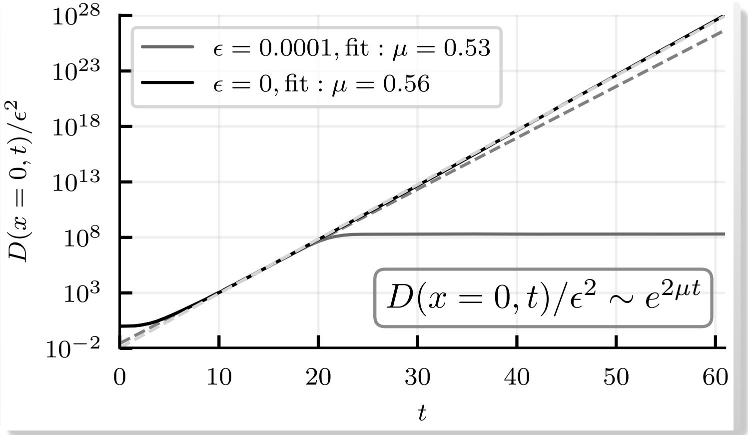
Isotropic spreading

- not symmetry protected

Exponential growth near front

- soliton-like motion

Exponential growth of decorrelator



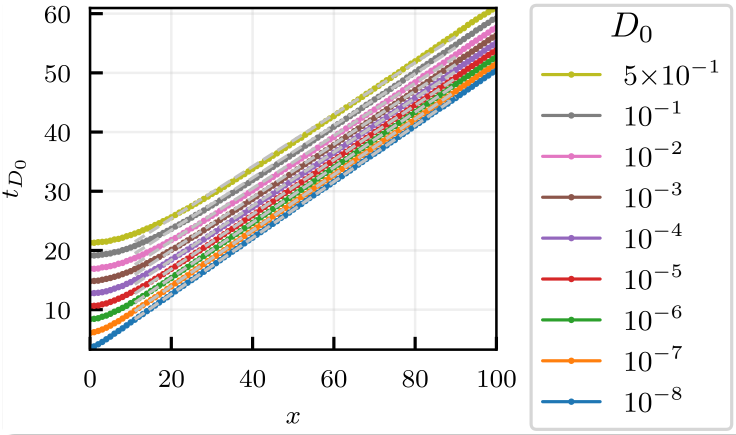
De-correlator $D(x=0,t) \sim e^{2\mu t}$

- defines Lyapunov exponent

Butterfly speed from ballistic propagation

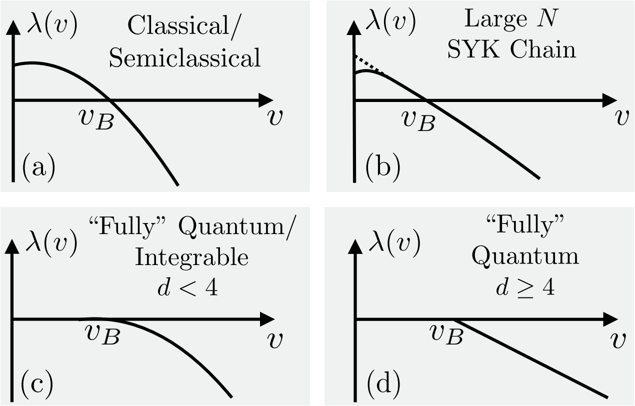
Arrival times $t(x)$ for which $D(x,t) > D_0$ for threshold D_0

- $t(x) = x/v_b$ defines butterfly speed v_b



Velocity dependent Lyapunov exponent

$$\mathrm{OTOC}(x,t) \sim e^{\lambda(v=x/t)t}$$

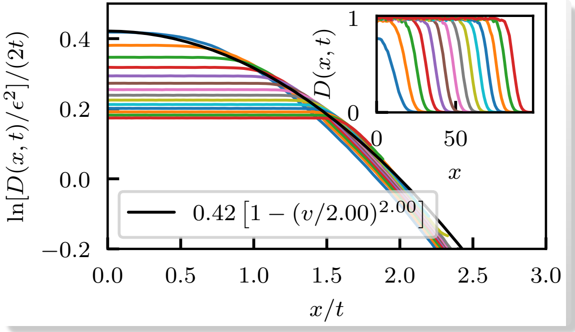


Nahum ’18

‘Soliton’ shape and Lyapunov exponent

Propagating front has robust shape

- cf. combustion equation Aleiner, Faoro, Ioffe



De-correlator exhibits scaling collapse along rays $v = x/t$

$$D(x,t) \sim \exp[2\mu(1-(v/v_b)^\nu)t]$$

velocity dependent Lyapunov

$$\lambda(v) = \mu(1 - v/v_b)^\nu$$

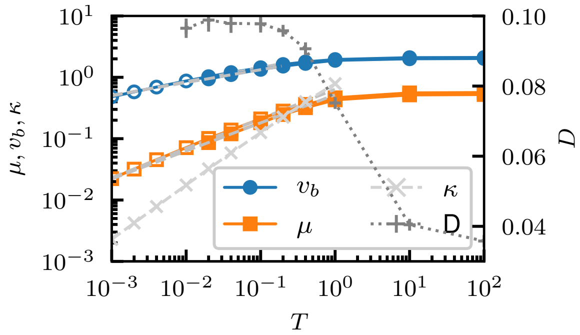
Temperature-dependence of chaos

'Microscopic'

- **Butterfly speed** v_b of ballistic propagation
- **Lyapunov exponent** μ of OTOC growth

'Macroscopic'

- **Spin-relaxation rate** κ of spin auto-correlation $A(t) \sim e^{-\kappa t}$
- **Spin diffusion constant** D from $\mathcal{S}(\mathbf{q}, \omega) \sim 1/[(Dq^2)^2 + \omega^2]$



at low T

$v_b \sim T^{0.25}$

$\mu \sim T^{0.5}$

$\kappa \sim T$

$D \sim \text{const}$

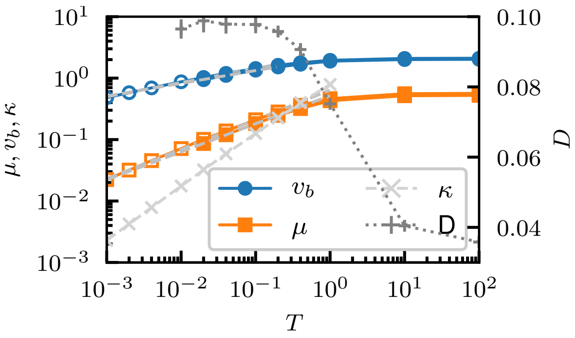
Temperature-dependence of chaos: power laws

Find power laws throughout

- some 'trivial', e.g. heat capacity $C \sim T^0$ from equipartition

Exchange constant drops out

- time reparametrisation invariance Kurchan, SYK, . . .



at low T

$v_b \sim T^{0.25}$

$\mu \sim T^{0.5}$

$\kappa \sim T$

$D \sim \mathrm{const}$

Obeys relation from holographics

$D \sim v_b^2/\mu$

- microscopics $\leftrightarrow$ transport

Chaos in a classical spin-liquid

Main results

- Spin-liquid ideal platform to study chaos for $T \in (0, \infty)$ ✓
- Powerlaw scaling of relevant quantities: μ, v_b, κ ✓
  - Connection between macroscopic transport and microscopic chaos $D \sim v_b^2/\mu$ ✓
  - $\mu \sim T^{0.5}$ vanishes slower than quantum bound ✓/?

Part II: Chaos across phase transitions

Q: Fundamental

- Effect of (thermal) phase transitions?
- Relevance of Symmetry-Breaking?
- Relevance of quasi-particles?

Q: specific

- How does the low T ordered state cross over into chaotic paramagnet?
- How do magnons affect the phenomenology discussed so far?

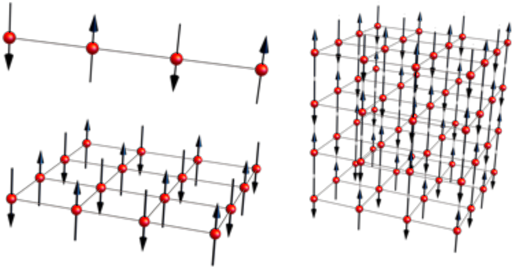
Heisenberg magnets on cubic lattices

Model

- Heisenberg O(3) spins

$$H = J \sum_{\langle \mathbf{i},\mathbf{j} \rangle} \mathbf{S}_i \cdot \mathbf{S}_j$$

- neighbouring spins interact
- FM ($J < 0$) / AFM ($J > 0$)



Classical FM/AFM

- fully ordered groundstate ($T = 0$)
- thermal phase transition at finite T ($d = 3$)
- finite systems order at finite T ($d = 1, 2$)

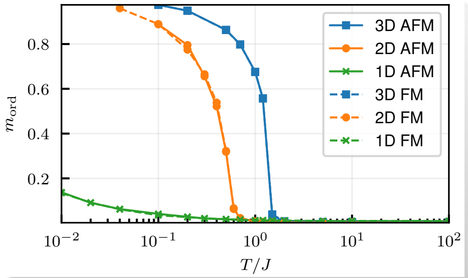
Semiclassical Dynamics

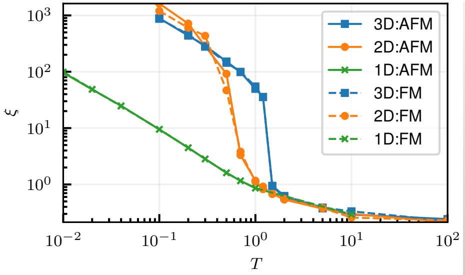
$$\frac{d\mathbf{S}_i(t)}{dt} = -\mathbf{S}_i(t) \times \sum_j J_{ij} \mathbf{S}_j(t)$$

Thermodynamics

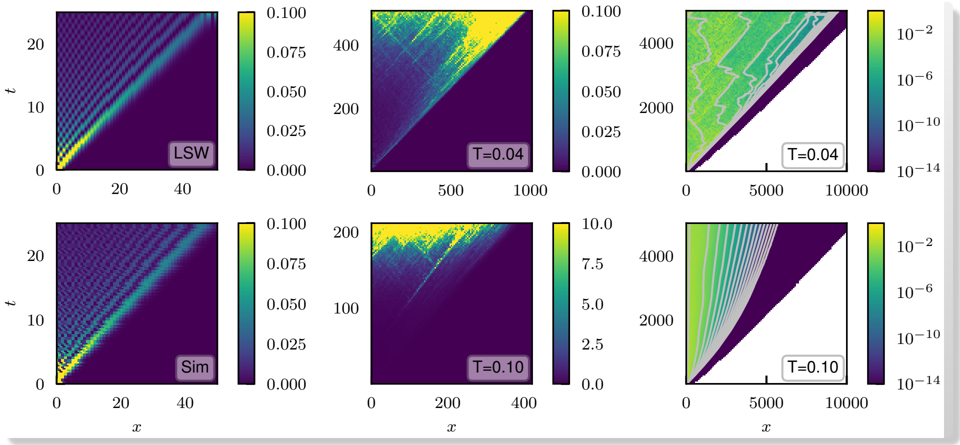
$$m_{ord} = \sum_i (-1)^i \langle \mathbf{S}_i(t) \rangle$$

$$\langle \mathbf{S}_i \cdot \mathbf{S}_j \rangle \sim e^{-|\mathbf{r}_i - \mathbf{r}_j|/\xi}$$

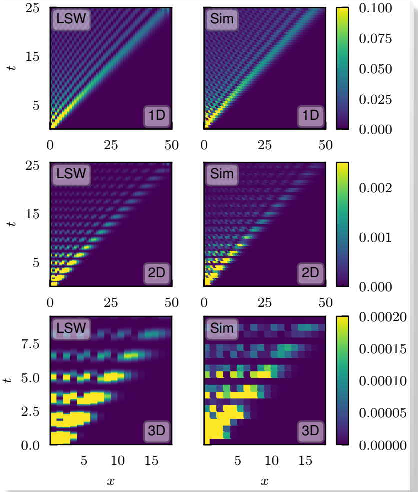


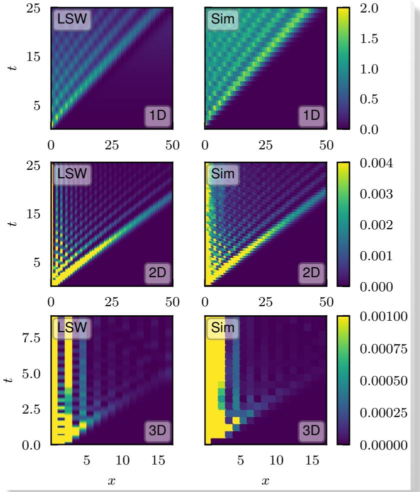


Regimes of the Decorrelator



Integrable Regime





LSW theory (see Phys. Rev. B 103, 174302)

- low T regime as spinwaves propagating in a fully ordered background

$$\mathbf{S}_i = \mathbf{n}_i \sqrt{1 - \mathbf{L}_i^2} + \mathbf{L}_i$$

- can expand equations of motion

$$\begin{aligned}
\partial_t \delta \mathbf{S}_i = & \mathbf{n}_i \times \sum_{j \in i} J_{ij} \delta \mathbf{S}_j + \delta \mathbf{S}_i \times \sum_{j \in i} J_{ij} \mathbf{n}_j \\
& + \mathbf{L}_i \times \sum_{j \in i} J_{ij} \delta \mathbf{S}_j + \delta \mathbf{S}_i \times \sum_{j \in i} J_{ij} \mathbf{L}_j \\
& - \frac{1}{2} \mathbf{L}_i^2 \mathbf{n}_i \times \sum_{j \in i} J_{ij} \delta \mathbf{S}_j - \frac{1}{2} \delta \mathbf{S}_i \times \sum_{j \in i} J_{ij} \mathbf{n}_j \, \mathbf{L}_j^2 \\
& + \delta \mathbf{S}_i \times \sum_{j \in i} J_{ij} \delta \mathbf{S}_j
\end{aligned}$$

- dynamics allows perturbative series/diagrammatics

$$\delta \mathbf{S}_{\mathbf{k}}(t) = \delta \mathbf{S}_{\mathbf{k}}^0(t) + G_{\mathbf{k}}^0(t) \sum_{\mathbf{q}} \int_0^t dt' \, \mathcal{A}_{\mathbf{k},\mathbf{q}}(t') \cdot \delta \mathbf{S}_{\mathbf{k}-\mathbf{q}}(t')$$

LSW Theory predictions (free FM Decorrelator)

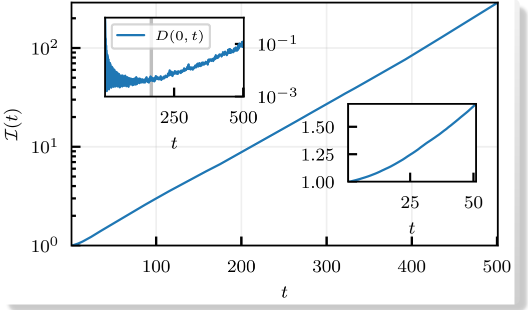
$$\boxed{\frac{\mathcal{D}(i,t)}{\varepsilon^2} = (1 - m_T^2)^2 \delta_{i,0} + m_T^2 \mathcal{F}_d(i,t)}$$

where m_T is the magnetisation and

$$\mathcal{F}_d(i,t) = \begin{cases} \left(J_{i_x}(2t)\right)^2 \ , & (d=1) \\ \left(J_{i_x}(2t)\right)^2 \left(J_{i_y}(2t)\right)^2 , & (d=2) \\ \left(J_{i_x}(2t)\right)^2 \left(J_{i_y}(2t)\right)^2 \left(J_{i_z}(2t)\right)^2 , & (d=3) \end{cases}$$

- Explicit form quantitatively correct in integrable regime
- $v_{LC} = 2\sqrt{d}$
- powerlaw decay t^{-d} with the light-cone
- $\mathcal{I}(t) = \sum_i \mathcal{D}(i,t) = const$

Onset of chaos



- $\mathcal{I}(t)$ weakly increasing for $t \lesssim 1/\lambda$
- $D(0, t)$ powerlaw decay for $t \lesssim 1/\lambda$
- long-time exponential chaos

Spin-wave life-time and Lyapunov

- Resum perturbative expansion

$$\delta\mathbf{S}_{\mathbf{k}}(t)\cdot\delta\mathbf{S}_{-\mathbf{k}}(t)=\epsilon^2+\frac{1}{N}\boldsymbol{\epsilon}^T\cdot\sum_{\mathbf{q}_1}\int_0^t dt_1\,\Big[\mathcal{A}_{-\mathbf{k};\mathbf{q}_1}(t_1)\times G^0_{-\mathbf{k}-\mathbf{q}_1}(t_1)+\left[G^0_{\mathbf{k}-\mathbf{q}_1}(t_1)\right]^T\left[\mathcal{A}_{\mathbf{k};\mathbf{q}_1}(t_1)\right]^T\Big]\cdot\boldsymbol{\epsilon}+\cdots$$

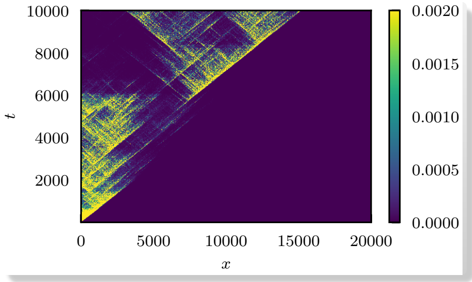
- to obtain

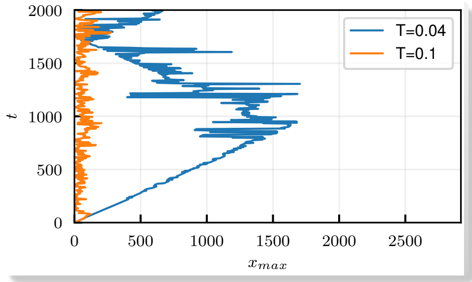
$$\delta\mathbf{S}_{\mathbf{k}}(t)\cdot\delta\mathbf{S}_{-\mathbf{k}}(t)\sim\exp(2t/\tau_{\mathbf{k}})$$

- This connects the spin-wave lifetine τ_k to the Lyapunov via

$$\lambda\sim\max_{\mathbf{k}}\frac{1}{\tau_{\mathbf{k}}}$$

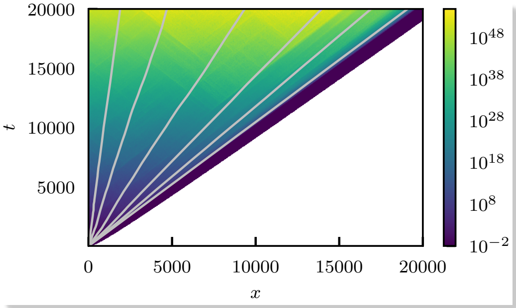
Scarred Regime





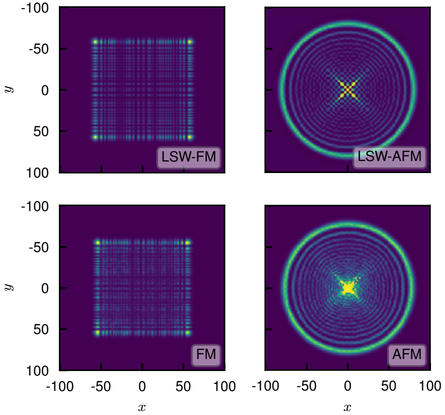
- Emergence of ballistically propagating excitations
- Particles scatter/get reflected
- scattering events proliferate/lead to chaos

Late-time chaos



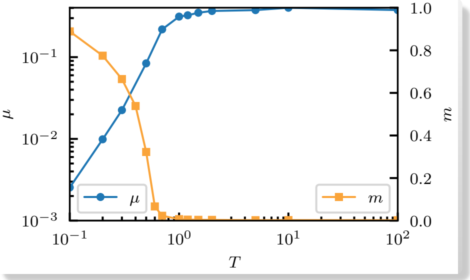
- Many overlapping light-cones/scattering events
- At long scales/times smooth appearance

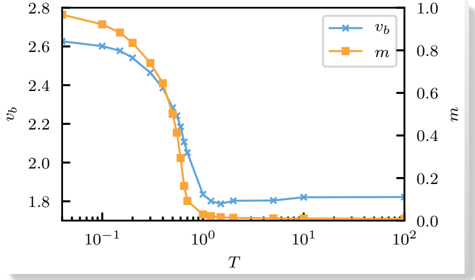
Wavefronts of the Decorrelator



- FM and AFM show different light-cones
- consistent with LSW theory

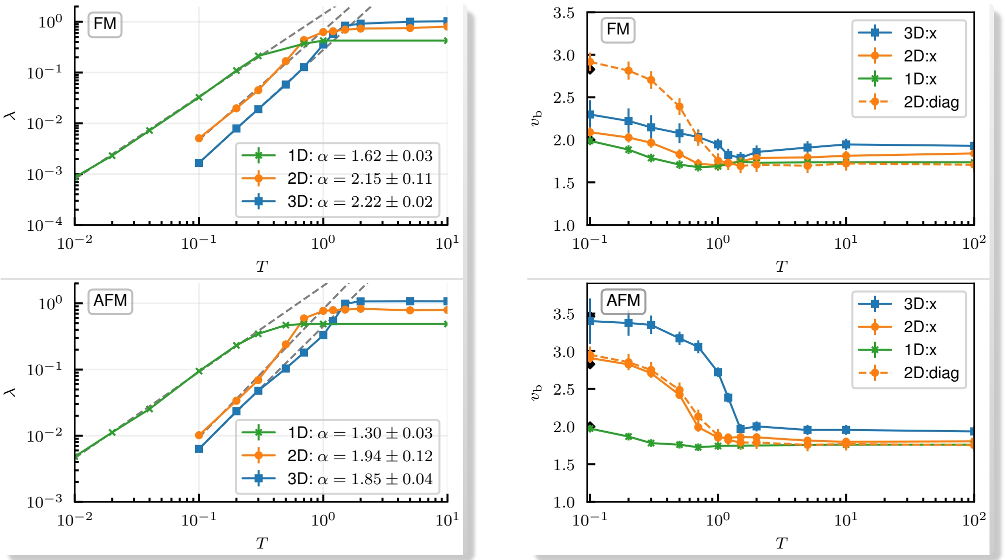
Behaviour across the phase transition I





- At transition lyapunov changes characteristically
- Emergence of Order/Magnons subsumes the butterfly velocity

Behaviour across the phase transition II



Chaos across phase transitions

Main results

- Phase transition leaves fingerprint in characteristics of chaos: v_b, λ
- Magnons subsume the butterfly speed in the ordered regime
- Magnon scattering results in emergence of chaos
- Magnon lifetime related to the Lyapunov exponent